QUANTIZED VORTICES IN ARBITRARY DIMENSIONS AND THE NORMAL-TO-SUPERFLUID PHASE TRANSITION

BY

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DISSERTATION

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Abstract

The structure and energetics of superflow around quantized vortices, and the motion inherited by these vortices from this superflow, are explored in the general setting of a superfluid in arbitrary dimensions. The vortices may be idealized as objects of co-dimension two, such as one-dimensional loops and two-dimensional closed surfaces, respectively, in the cases of three- and four-dimensional superfluidity. By using the analogy between vortical superflow and Ampère-Maxwell magnetostatics, the equilibrium superflow containing any specified collection of vortices is constructed. The energy of the superflow is found to take on a simple form for vortices that are smooth and asymptotically large, compared with the vortex core size. The motion of vortices is analyzed in general, as well as for the special cases of hyper-spherical and weakly distorted hyper-planar vortices. In all dimensions, vortex motion reflects vortex geometry. In dimension four and higher, this includes not only extrinsic but also intrinsic aspects of the vortex shape, which enter via the first and second fundamental forms of classical geometry. For hyper-spherical vortices, which generalize the vortex rings of three dimensional superfluidity, the energy-momentum relation is determined. Simple scaling arguments recover the essential features of these results, up to numerical and logarithmic factors. Extending these results to systems containing multiple vortices is elementary due to the linearity of the theory. The energy for multiple vortices is thus a sum of self-energies and power-law interaction terms. The statistical mechanics of a system containing vortices is addressed via the grand canonical partition function. A renormalization-group analysis in which the low energy excitations are integrated approximately, is used to compute certain critical coefficients. The exponents obtained via this approximate procedure are compared

with values obtained previously by other means. For dimensions higher than three the superfluid density is found to vanish as the critical temperature is approached from below.

To the memory of my mother.

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Chapter 1

Introduction

First conceived of and analyzed by Onsager [30, 40] and Feynman [9, 7], quantized vortices are topologically stable excitations of superfluid helium-four [24]. They are manifestations of quantum mechanics which owe their stability to the quantization law that the circulation of superflow obeys, leaving their imprint at arbitrarily large distances. Vortex excitations play an essential role in determining the equilibrium properties of superfluid helium-four, especially in the mechanism via which superfluidity is lost as the temperature is raised through the “lambda” transition [23] to the normal fluid state. They also play an essential role in the mechanism via which superflow in toroid-shaped samples is dissipated. For an in-depth account of the properties of quantized vortices, see Ref. [6].

When the space filled by the superfluid helium-four (i.e., the ambient space) is two- or three-dimensional, and when they are viewed at length-scales larger than their core size, quantized vortices are, respectively, the familiar, geometrical point-like vortices of two-dimensional superfluidity or line-like vortices of three-dimensional superfluidity. Motivated in part by the issue of the impact of topological excitations on critical phenomena in superfluid helium-four, which constitutes the second component of this thesis, the initial aim of the dissertation is to examine some of the characteristic properties of quantized vortices that are familiar in two- and three-dimensional superfluid helium-four, and to determine how these topological excitations and their characteristic properties can be extended to situations in which the superfluid (or any system characterized by broken global $U(1)$ symmetry) fills spaces having arbitrary numbers of dimensions. For the case of three dimensions, these properties are analyzed in Ref. [21]. Properties that we shall address here include the structure of quantized vortices and the flows around them, the motion that vortices inherit from such flows (which we shall refer to as dynamics), and the flow energetics. We shall assume that the vortices move with the superfluid velocity, that the vortices themselves have no mass, and that spectral flow is neglected. As these assumptions are believed to be reasonable in two and three dimensions [33], and we see no reason for them to be peculiar to any particular spatial dimension, we shall assume that they hold in higher dimensions, too. In all dimensions, these excitations (again, when viewed at length-scales larger than their core size) continue to be geometrical structures having codimension two: their dimensionality grows with the dimension of the ambient space, always remaining two dimensions behind. Nevertheless, we shall use the terminology "quantized vortex" (or, for simplicity, just "vortex") for such excitations in all dimensions, not only two and three.

We have chosen to present our developments [8, 12] using the traditional language of vector and tensor calculus. However, along the way we shall pause to mention how these developments appear when formulated in the language of exterior calculus and differential forms; see Refs. [35, 11]. This language is in fact tailor-made for the questions at hand, and enables a highly economical development. However, we have not relied on it, as there may be readers for whom it would require an unfamiliar level of abstraction.

This dissertation is organized as follows. We begin with a brief introduction of systems that exhibit $U(1)$ symmetry (the XY spin model, and two dimensional superfluidity), showing key features such as the mechanism of the Kosterlitz-Thouless. In Chapter 3, we discuss the structure of vortices in a superfluid in arbitrary dimensions of space, first from the perspective of length-scales and topology, and then by developing the conditions that flows of a specific vortex content must obey, in both the integral and the differential versions. In Section 3.3, we introduce considerations of kinetic energy, examining superflows of given vortical content at equilibrium, and solving for the associated velocity fields of these flows. In Section 3.4, we analyze the dynamics that vortices inherit from the equilibrium flows associated with them. Here, we specialize to asymptotically large, smooth vortices, for which the dynamics turns out to be a simple and clear reflection of the intrinsic and extrinsic geometry of the shapes of the vortices. To illustrate our results, we consider the velocities of spherical and hyper-spherical vortices, as well as the small-amplitude excitations of planar and hyper-planar vortices. In Section 3.5, we return to issues of energetics, and determine the equilibrium kinetic energy of superflows containing individual vortices in terms of the geometry of the shapes of the vortices. We illustrate this result for large, smooth—but otherwise arbitrarily shaped—vortices, for which a simple, asymptotic formula holds, and we give an elementary argument for this asymptotic formula. Here, we also address the energetics of maximally symmetrically shaped spherical and hyper-spherical vortices, and we use our knowledge of the energies and velocities of the spherical and hyper-spherical vortices to determine their momenta and, hence, their energy-momentum relations. In Section 3.7, we use scaling and dimensional analysis to obtain the aforementioned results, up to numerical factors and logarithmic dependencies on the short-distance cut-off, i.e., the core size of the vortices. The results reported in Chapter 3 were published in Ref [8].

In Chapter 4 we examine the superfluid-normal transition of systems that have $U(1)$ symmetry. Starting from the total kinetic energy of the superflow obtained in Section 4.1.1 we derive the Hamiltonian for a collection of vortices. The Hamiltonian can be separated into two distinct parts: self-energies of the vortices and interaction between vortices. In the limiting case of the distances between vortices being much larger than their linear dimensions, interactions are inversely proportional to the distance between the vortices raised to a power that is dependent on the dimension of the embedding space.

In the next section, Section 4 we begin an examination of the statistical mechanics of a $U(1)$ system in arbitrary dimension of space viewed from the perspective of the grand canonical ensemble of vortices. To do this, we examine the partition function for a fluctuating number of vortices of arbitrary shape and location. We define two coupling parameters, the superfluid stiffness and and the fugacity of a vortex, and we carry out a renormalization rroup analysis to construct the flow equations for these couplings. In the ususal way linearization the flow equations near a fixed point allows us to compute the relevant/irrelevant axes in the coupling space and, via a scaling argument, the critical exponents ν and α governing the correlation length dependence and specific heat dependence of the reduced temperature.

We end, in Chapter 5, with some concluding remarks.

We have tried to streamline the presentation by relegating to a sequence of appendices much of the technical material.

Chapter 2

Systems that exhibit $U(1)$ symmetry

2.1 XY model

We begin this chapter with an introductory account [13, 26, 31] of the one of the simplest models that exhibits $U(1)$ symmetry, namely the XY-planar model of magnetism. Let us consider a D-dimensional square lattice of lattice spacing b inhabited by spins that rotate in the (x, y) plane located at the sites of the lattice. The Hamiltonian $\mathcal{H}$ for this model is given by:

$$\mathcal{H} = -\frac{1}{2} J \sum_{\langle i,j \rangle} \vec{s}_i \cdot \vec{s}_j, \tag{2.1}$$

where J is a measure of the interaction strength between the spins, i and j index lattice sites and the sum is over nearest neighbor pairs. We can equivalently parametrize the spins either by their components in the plane, $\vec{s} = (s\cos\theta, s\sin\theta)$ or by a complex order parameter Ψ, defined via a magnitude $|\Psi|$ and a phase θ, i.e., $\Psi = |\Psi|e^{i\theta}$. The variable θ is the angle between the spin direction and an arbitrarily chosen axis and is a continuous periodic variable.

In the low-temperature regime, where large fluctuations of the spin orientations are thermodynamically suppressed – only slowly varying configurations give significant contributions to the partition function – we can expand Eq. (2.1) to second order in the phase difference between the nearest neighbors to obtaing:

$$\mathcal{H} \approx -nNJ + \frac{1}{4} J \sum_{\langle i,j \rangle} (\theta_i - \theta_j)^2 + \cdots \approx -nNJ + \frac{1}{2} J b^D \sum_i |\vec{\nabla}\theta_i|^2 + \cdots, \tag{2.2}$$

where N is the total number of lattice sites, n is the number of nearest neighbors per lattice site, and the summation is understood to occur over all nearest neighbor pairs.

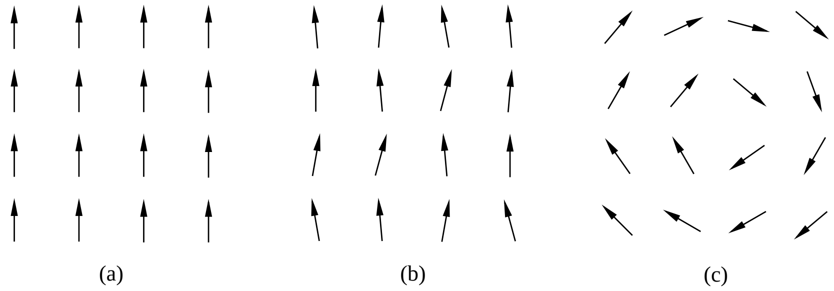

Figure 2.1: Spins in a two dimensional XY model. (a) Ground state, (b) spin wave excitation, (c) vortex excitation

The perfectly ordered ground state is a state is a state in which the spin orientation is spatially uniform. "Spin wave" excitations are states where the phase of the order parameter θ fluctuates around the ground state value. Another possible excitation above the ground state is a vortex, shown in Fig. 2.1. On any line integral along a closed path around the center of the vortex, the phase winds by a factor of $2\pi k$, where k is an integer called the strength of the vortex.

For a vortex of strength k, the phase θ is equal to the polar angle in the plane, multiplied by the strength, i.e. $\theta = \phi k$; the magnitude of its gradient is equal to the strength divided by the radius, $|\vec{\nabla}\theta_i| = k/r$. Using the continuum version of the Hamiltonian in Eq. (2.2), where we have omitted the (constant) ground state energy, i.e.,

$$\mathcal{H} = \frac{1}{2} J \int d^2r |\vec{\nabla}\theta_i|^2, \tag{2.3}$$

we can thus obtain the energy of a single vortex excitation:

$$E_{vortex} = \pi J k^2 \ln \frac{R}{\xi}, \tag{2.4}$$

where R is the size of the system and ξ is a short-distance cut off equal to the size of the vortex core.

In the high-temperature equilibrium state there is no long range order, and spin-spin correlations decay exponentially with separation. In two dimensions, however, there exists a "quasi"-ordered phase in the low temperature regime, in which correlations decay algebraically with separation rather than exponentially [22]. The transition between these two phases occurs when creation of vortices that are unbound from anti-vortices becomes thermally favorable, which has the effect of destroying the long-range order.

A beautifully simple argument, developed by Kosterlitz and Thouless [18], explains the role played by the vortices in the transition from the ordered state the to disordered state. The energy of a single superfluid vortex obtained in Eq. (2.4) is given by $\frac{1}{2}\rho(2\pi)\ln(R/\xi)^2$, where R is the linear dimension of the sample and ξ is a short distance cut-off, identified as the vortex core size. The vortex has an entropy associated with all possible locations in the sample given by $k_B \ln(R/\xi)^2$, where k_B is Boltzman's constant. The free energy is then given by

$$F = E - TS \approx (\pi\rho - 2k_B T)\ln(R/\xi) \tag{2.5}$$

and has a critical temperature equal to $T = \pi\rho/2k_B$. We can distinguish two cases: in the first occuring for $T < \pi\rho/2k_B$, the free energy has a minimum when there are no vortices; in the second case, for $T > \pi\rho/2k_B$, the free energy has a minimum when vortices proliferate. Kosterlitz and Thouless therefore reasoned that the critical temperature is a transition temperature as vortices destroy the long-range order.

2.2 Superfluid helium films

In the continuum limit, the XY-model is used to model a range of physical systems that are characterized by the same $U(1)$ symmetry (e.g., planar magnets, certain liquid crystal systems, and superfluid films lie in this category). In this section we discuss properties characteristic to superfluid helium films. Theoretical calculations give remarkably accurate predictions [34] for the values of the critical exponents as well as the existence of a universal jump in the superfluid density at the critical point [29].

We begin by denoting the superfluid velocity for any point $\vec{x}$ in the mass of the fluid by $\vec{v}(x)$. This velocity can be decomposed into two components [20]:

$$\vec{v} = \vec{v}_{\parallel} + \vec{v}_{\perp}, \tag{2.6}$$

where $\vec{v}_{\parallel}$ is a smooth component that satisfies $\vec{\nabla} \times \vec{v}_{\parallel} = 0$ thus we can be written as the divergence of a field ϕ, i.e., $\vec{v}_{\parallel} = \vec{\nabla}\phi$. The second component, $\vec{v}_{\perp}$, arises from vortices and is divergence free (i. e. $\vec{\nabla} \cdot \vec{v}_{\perp}$).

For an incompressible fluid the Hamiltonian is given by the kinetic energy:

$$\mathcal{H} = \frac{1}{2}\rho \int d^2x \, |\vec{v}_{\parallel} + \vec{v}_{\perp}|^2, \tag{2.7}$$

where ρ is the superfluid mass density.

We define the superfluid stiffness K by the quantity

$$K = \rho / k_B T, \tag{2.8}$$

and we henceforth use it in the Hamiltonian, instead of the density, as it is the quantity that appears in the partition function. Expanding the Hamiltonian from Eq. (2.7),

$$\mathcal{H} = \frac{1}{2}\rho \int d^2x \, (\vec{v}_{\parallel}^{\,2} + \vec{v}_{\perp}^{\,2} + 2\vec{v}_{\parallel} \cdot \vec{v}_{\perp}) \tag{2.9}$$

we can interpret the terms as follows: the first is the energy of the flow if no vortices were present, the second term would be the Hamiltonian if only the vortices alone were present, and the third term we treat as a small perturbation. Up to second order in v the free energy is

$$F = F_0 + \frac{1}{2}K\Omega v^2 - \frac{1}{2}K^2\Omega v^2 \int d^2x \langle \vec{v}_\perp(x) \cdot \vec{v}_\perp(0)\rangle + \cdots , \tag{2.10}$$

where Ω is the volume of the sample and $\langle \vec{v}_\perp(x)\, \vec{v}_\perp(0)\rangle$ is the velocity-velocity correlation function. Thus, from Eq. (2.10) we can define the renormalized stiffness K_R:

$$K_R = K - \frac{1}{2}K^2 \int d^2x \langle \vec{v}_\perp(x) \cdot \vec{v}_\perp(0)\rangle. \tag{2.11}$$

We now introduce a configuration of vortices (i.e., vortices located at position x_α in the superfluid), represented by the vortex density $n_v(x)$:

$$\vec{\nabla} \times \vec{v}_\perp = 2\pi n_v(x)\hat{z} = 2\pi\hat{z}\sum_\alpha \delta(x - x_\alpha). \tag{2.12}$$

By using the Fourier representation for the velocity we can relate the velocity-velocity correlation function to the vortex density correlation function. We state the result given in Ref. [19]:

$$\int d^2x\, \langle \vec{v}_\perp(x) \cdot \vec{v}_\perp(0)\rangle = 2\pi \int d^2x\, x^2\, \langle n_v(x)\, n_v(0)\rangle. \tag{2.13}$$

The density correlation function can be computed in the following way: two vortices of opposite vorticity are present in the superfluid with a probability proportional to the fugacity (i.e., the energy cost of producing a vortex), and, in accordance with Eq. (2.4), attract each other via a logarithmic potential $E = 2\pi \ln(r/\xi)$. Hence the density correlation function is given by:

$$\langle n(0)\, n(r)\rangle = -2y^2 e^{-2\pi K \ln\frac{r}{\xi}}. \tag{2.14}$$

Because we are assuming that the fugacity y is a small parameter, i.e., $y \ll 1$, we can express the renormalized stiffness K_R in terms of the "bare" stiffness K as

$$\frac{1}{K_R} = \frac{1}{K} + 2\pi^3 y(l)^2 \int_\xi^\infty \frac{dr}{\xi}\left(\frac{r}{\xi}\right)^{3-2\pi K(l)} + \mathcal{O}(y^4(l)). \tag{2.15}$$

This expression indicates that starting with the microscopic K and y, we have to take into account the total reduction in energy due to polarization of the vortex pairs.

The integral is convergent only at low temperatures, i.e., $K > 2/\pi$, but one can understand this result more completely using a renormalization-group approach[15]. We allow the parameters K and y to depend on a scaling parameter l, and evaluate the integral piece-by-piece from $(\xi, \xi(1+dl))$ and $(\xi(1+dl), \infty)$:

$$\int_{\xi}^{\infty} \cdots = \int_{\xi}^{\xi(1+dl)} \cdots + \int_{\xi(1+dl)}^{\infty} \cdots . \tag{2.16}$$

The first integral can be recasted into a new K and the second integral can be put in the same form as before with the newly defined short distance cut-off $\xi(1+dl)$ by changing y. These transformations do not modify the renormalized value K_R, and we can thus define the following recursive relations for $K(l)$ and $y(l)$:

$$\frac{d}{dl}K^{-1}(l) = 2\pi^2 y^2(l); \tag{2.17}$$

$$\frac{d}{dl}y(l) = (2 - \pi K)y(l). \tag{2.18}$$

In Fig. 4.1 we represent the flow of the couplings K and y obtained by iterating Eq. (2.18).

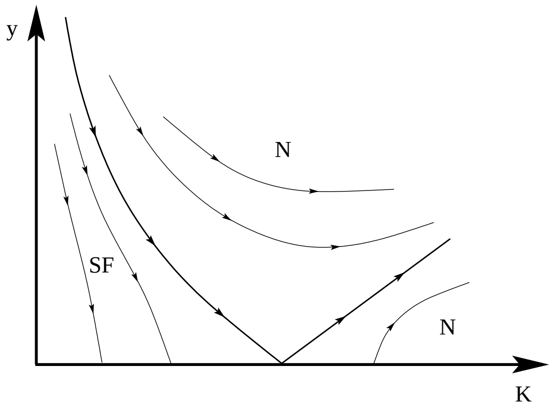

Figure 2.2: Renormalization flows in the (K, y)-plane. S_1 and S_2 are separatices dividing the plane into three regions. SF is the superfluid region; N is the normal sregion.

There are two separation curves intersecting each other at the critical point $K = 2/\pi$. We note that in the superfluid region (i.e., SF) the fugacity iterates to small values, whereas in the normal region (i.e. N) y iterates to large values, which lie outside the perturbative regime that we had assumed.

The calculation of K_R is difficult if one is trying to use Eq. (2.15); instead. we note that close to the critical point the fugacity vanishes asymptotically (i.e., $\lim_{l\to\infty} y(l) = 0$). Across the critical point the renormalized value of the coupling K_R has an universal jump [29] of magnitude $\pi/2$, a feature that had been verified experimentally [2] and is characteristic of only to superfluid helium films. In contrast, in three-dimensional helium no discontinuity is observed for the superfluid density at the critical point. Instead the density vanishes continuously at the transition temperature.

In the following chapters we shall attempt to generalize concepts such as superflow, vortices, their dynamics, and their statistical mechanics to an arbitrary number of dimensions and, hence to study the normal-to-superfluid transition.

Chapter 3

Structure of superfluid vortices in arbitrary dimensions

3.1 General morphology of vortices

We consider a large, D-dimensional, Euclidian, hyper-cubic volume Ω of superfluid, with each point in Ω being specified by its D-component position-vector x, the Cartesian components of which we denote by $\{x_d\}_{d=1}^{D}$. On the faces of Ω we suppose that all fields obey periodic boundary conditions. We shall be concerned with flows of superfluid in Ω, and these we describe via their D-vector flow fields $\mathcal{V}(x)$. The phenomena that we shall be addressing occur on length-scales larger than the characteristic superfluid coherence- or healing-length ξ, which determines the length-scale over which the superfluid density falls to zero as the centre of a vortex is approached (i.e., the vortex core size).

Provided we do indeed examine flows on length-scales larger than ξ, and hence restrict our attention to long-wavelength, low-energy excitations of the superfluid, the *amplitude* $|\Psi(x)|$ of the complex scalar superfluid order parameter field $\Psi(x)$ can be taken to be a nonzero constant $|\Psi_0|$ away from the vortices. In addition, these vortices can be idealized as being supported on vanishingly thin sub-manifolds of the ambient D-dimensional space. The remaining freedom lies in the *phase factor* $\Psi(x)/|\Psi(x)|$ of $\Psi(x)$, and—except on vortex sub-manifolds, where it is not defined—this takes values on the unit circle in the complex plane. Owing to the topology of the coset space (in this case, the unit circle) in which this phase factor resides, and specifically the fact that its fundamental homotopy group Π_1 is the additive group $\mathbb{Z}$ of the integers $n = 0, \pm 1, \ldots$, these vortices are indeed topologically stable defects in the superfluid order [27]. The *strength* n of any vortex is the number of times the phase factor winds through 2π as the vortex is encircled a single time. For D-dimensional superfluids, and at the length-scales of interest, these vortices are supported on manifolds of dimension $D-2$, a combination that will occur frequently, so we shall denote it by $\not{D}$. Said another way, the vortices reside on co-dimension-2 sub-manifolds of D-dimensional space, which means that we would have to add two further dimensions to "fatten" each vortex into a space-filling object. In any dimension $D \geq 2$, there are one-dimensional closed paths that loop around the vortices. As we shall discuss shortly, the circulation of the flow around such loops is quantized and nonzero, whereas the circulation around closed paths that do not loop around a vortex is zero.

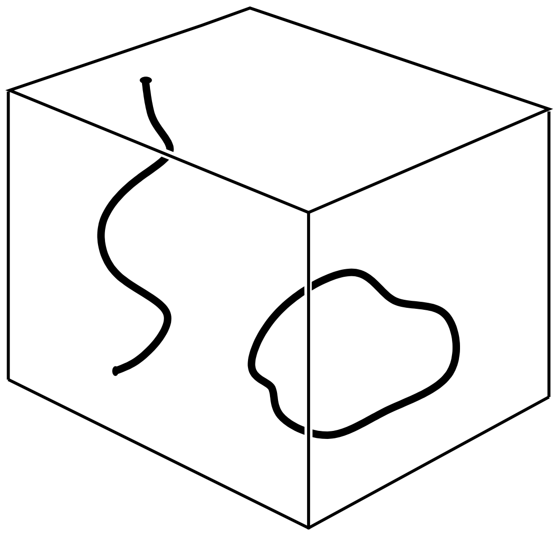

Figure 3.1: Vortices in three-dimensional superfluids. They form closed one-dimensional loops, possibly via the periodic boundary conditions.

In the familiar case of superfluids in three dimensions, the vortices are geometrical objects that are supported on one-dimensional sub-manifolds (see Fig. 3.1). The loops that encircle the vortices, along which the circulation of the flow is quantized, are also one-dimensional objects [1]. In the equally familiar case of superfluids in $D = 2$, each vortex is a pair of opposing-circulation zero-dimensional objects (i.e., a structure that it usually called a

[1]The vortices themselves are closed loops that may or may not wind through the boundaries of the sample, a possibility allowed by the D-torus topology of the sample. This topology arises from the adopted periodic boundary conditions but will not play a significant role here.

vortex/anti-vortex pair). The loops around one or other member of the pair, however, along which the circulation of the flow is quantized, remain one dimensional. (Strictly speaking, in $D = 2$ each vortex is supported on a pair of points, which is not a manifold, but we shall admit this abuse of nomenclature.)

Our main objectives are to explore the spatial structure and energetics of flows possessing a set of vortices at prescribed locations, as well as the motion that such flows confer on the vortices themselves. The flows that we shall consider are states of thermodynamic equilibrium, but are maintained away from the no-flow state, $\mathcal{V} = 0$, because they are constrained to have a prescribed vortical content.

3.2 Fixing the vortical content of superflows

The *circulation* κ of the flow along any one-dimensional closed contour is defined to be the line-integral of the corresponding velocity field along that contour:

$$\kappa := \oint \mathcal{V}_d(x)\, dx_d\,. \tag{3.1}$$

Here and throughout this paper, a summation from 1 to D is implied over repeated indices such as d, d_1, etc. The vortical content of the flow can then be prescribed via the following condition. For a vortex of strength n, the circulation κ is quantized according to the number of times $\mathcal{N}$ the contour encircles the $\not{D}$ [i.e., $(D-2)$]-dimensional sub-manifold supporting the vortex:

$$\kappa = (2\pi\hbar/M)\mathcal{N}n, \tag{3.2}$$

where $\hbar$ is Planck's constant and M is the mass of one of the particles whose condensation causes the superfluidity. We adopt units in which $\hbar/M$ has the value unity. In the language of differential forms, the flow field is described by the 1-form $\mathcal{V}(x)$, and the quantization of circulation (3.2) is expressed as

$$\oint \mathcal{V} = 2\pi \mathcal{N} n. \tag{3.3}$$

We shall be concerned with vortices of unit strength (i.e., $n = 1$). Owing to the linearity of the theory, results for higher-strength vortices can be straightforwardly obtained from those for unit-strength vortices via elementary scalings with n (i.e., velocities scale linearly with n; energies scale quadratically).

For a flow $\mathcal{V}$ to have the prescribed vortical content, it must be expressible as a sum of two contributions:

1. a real vector field $V(x)$ that is singular on any vortex-supporting sub-manifolds, and has the appropriate quantized circulation around them, and thus its contribution to the flow includes the flow's vortical part; and

2. the gradient of a real, scalar phase field $\Phi(x)$ that is smooth and *single-valued* throughout Ω. Owing to the latter property, this component of the flow is capable of contributing only to the irrotational (i.e., circulation-free) aspects of the flow.

Thus, we consider flows of the form

$$\mathcal{V}_d(x) = V_d(x) + \nabla_d \, \Phi(x), \tag{3.4}$$

in which the gradient ∇_d denotes the partial derivative $\partial/\partial x_d$ with respect to the d^{th} spatial coordinate x_d. (The $\nabla_d \, \Phi$ term carries a dimensional factor of $\hbar/M$, which we set to unity, earlier.) In the language of differential forms, Eq. (3.4) is expressed as

$$\mathcal{V} = V + d\Phi, \tag{3.5}$$

where $d\Phi$ is the exact 1-form that results from taking the exterior derivative of the 0-form (i.e., the function) Φ.

To ensure that the vortical part V of the flow $\mathcal{V}$ has the appropriate circulation on loops surrounding any vortex-supporting sub-manifolds, it must obey the following inhomogeneous partial differential equation:

$$\epsilon_{d_1\cdots d_D}\nabla_{d_{D-1}}V_{d_D}(x) = 2\pi J_{d_1\cdots d_{\not D}}(x), \tag{3.6}$$

where $\epsilon_{d_1\cdots d_D}$ is the completely skew-symmetric D-dimensional Levi-Civita symbol. Equation (3.6) is a natural generalization to D-dimensions of the static version of the three-dimensional differential expression of the Ampère-Maxwell law relating the magnetic field (the analogue of V) to the electric current density (the analogue of J), which in suitable units reads

$$\operatorname{curl}\mathbf{V} = 2\pi\mathbf{J}. \tag{3.7}$$

An account of the analogy between superfluidity and magnetostatics for the case of three dimensions is given in Ref. [17]. An extension of electrodynamics, which allows for extended, string-like sources and necessitates the introduction of rank-2 skew-symmetric gauge potentials, was developed by Kalb and Ramond [16] in the context of string theory. A more general extension, to settings involving arbitrary space-time dimensions and $p(>1)$-form gauge potentials, has been discussed by Teitelboim [37] and Henneaux and Teitelboim [14].

The source field in Eq. (3.6), J, is a completely skew-symmetric, rank $\not D$ tensor field, which obeys the divergencelessness condition[2] $\partial_{d_1}J_{d_1\cdots d_{\not D}}(x) = 0$, i.e., the consistency condition that follows, e.g., from the application of ∂_{d_1} to Eq. (3.6). (We discuss this point further in A.) This field J encodes the singularity in the density of the vorticity associated with any $\not D$-dimensional vortices in the superflow. The factor of 2π in Eq. (3.6) is extracted to ensure that unit-strength sources gives rise to unit quanta of circulation. That V must obey Eq. (3.6) follows from the integral form of this law, Eq. (3.2), which can be obtained

[2] We have chosen to use the term *divergenceless* for fields obeying the associated condition, but we could equally well have chosen to use instead *divergence free*, *solendoidal* or *transverse*.

from Eq. (3.6) via the D-dimensional version of Stokes' theorem. The right-hand side of Eq. (3.2) is the flux of the source $J(x)$ through the corresponding closed contour. (In its integral form, the three-dimensional Ampère-Maxwell law dictates that the line-integral of the magnetic field along a loop in three-dimensional space has a value determined by the flux of the electric current threading the loop.) The skew-symmetric tensor nature of the source field J, present for $D \geq 4$, is a natural generalization of the vectorial nature of J present in $D = 3$, and reflects the corresponding higher-than-one-dimensional geometry of the sub-manifold that supports the vortex, as we shall discuss shortly.

In the language of differential forms, Eq. (3.6) is expressed as an equality of $\not{D}$-forms, i.e.,

$$\star\, dV = 2\pi J, \tag{3.8}$$

where dV is the 2-form that results from taking the exterior derivative of V, the form $\star\, dV$ is the $\not{D}$-form dual to it that results from applying the (Eucidian) Hodge-$\star$ operator to it, and J is the source $\not{D}$-form. The dual of this equation, which is an equality of 2-forms obtained by applying $\star$ to it, reads

$$dV = \star\,(2\pi J). \tag{3.9}$$

In the language of differential forms, the divergencelessness of the source J, mentioned shortly after Eq. (3.7), amounts to J being a closed form, i.e., $dJ = 0$.

What value should the source field J have if the D-dimensional flow is to contain a single, unit-strength, quantized vortex? Recall that for $D = 3$ (so that $\not{D} = 1$), the vortex is one-dimensional and supported on the curve $\mathbf{x} = \mathbf{R}(s_1)$, parameterized by the single variable s_1 (see Fig. 3.2). In this case, the source takes the value

$$\mathbf{J}(\mathbf{x}) = \int ds_1 \, \frac{d\mathbf{R}}{ds_1} \, \delta^{(3)}(\mathbf{x} - \mathbf{R}(s_1)) \, , \tag{3.10}$$

associated with the analog of a unit-strength electrical current flowing along the curve and supported on it. The sign of this analogue of the current (i.e., its direction around the loop) determines the sense in which the superflow swirls around the vortex core, via a right-hand rule. Observe that the geometry of the line enters not only in terms of the curve $\mathbf{R}(s_1)$ on

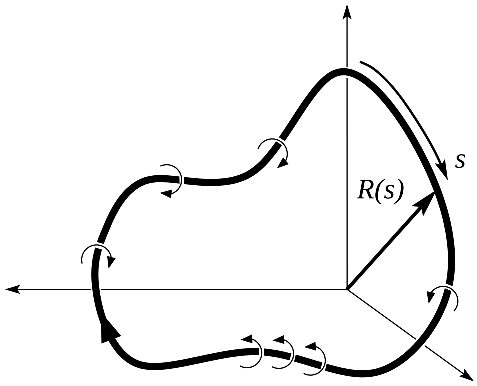

Figure 3.2: A loop vortex in a three-dimensional superfluid specified by $R(s)$ with arc-length parameter s, showing the direction of the source "electrical current" along the loop and the sense of the superflow encircling the loop.

which the vortex is supported, but also via the tangent vector to this curve, $d\mathbf{R}/ds_1$. The geometrically natural extension of the source to arbitrary dimension D involves:

1. concentrating the source on the appropriate $\not{D}$-dimensional sub-manifold;
2. specifying the shape of this sub-manifold, via $x = R(s)$, where s denotes the set of the $\not{D}$ variables $\{s_1, s_2, \ldots, s_{\not{D}}\}$ that, when they explore their domains, trace out the idealized core of the vortex; and
3. endowing the source with appropriate geometrical content, via the natural generalization of the tangent vector.

This generalization is provided by the oriented, $\not{D}$-dimensional (tensorial) volume

$$\varepsilon_{a_1 \cdots a_{\not{D}}}(\partial_{a_1} R_{d_1}) \cdots (\partial_{a_{\not{D}}} R_{d_{\not{D}}})$$

(where ∂_a denotes the partial derivative with respect to the a^{th} independent variable that parameterizes the vortex sub-manifold, in this case, $\partial/\partial s_a$) of the parallelepiped constructed from $\{\partial R/\partial s_a\}_{a=1}^{\not{D}}$, i.e., the set of $\not{D}$ linearly independent, coordinate tangent-vectors to the sub-manifold at $R(s)$. Together with the measure $d^{\not{D}}s$, this volume plays the role of an infinitesimal segment of vortex line (in $D = 3$) or patch of vortex area (in $D = 4$, as sketched in Fig. 3.3). Here, $\varepsilon_{a_1 \cdots a_{\not{D}}}$ is the completely skew-symmetric $\not{D}$-dimensional Levi-

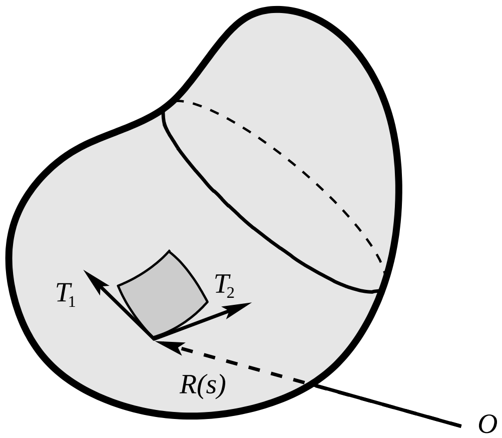

Figure 3.3: A two-dimensional vortex sub-manifold in a four-dimensional superfluid. T_1 $[= \partial_1 R(s)]$ and T_2 $[= \partial_2 R(s)]$ are two coordinate tangent vectors to the sub-manifold at the point $R(s)$. These tangent vectors define a "patch" of the sub-manifold.

Civita symbol, in contrast with the D-dimensional $\epsilon_{d_1 \cdots d_D}$, which we defined shortly after Eq. (3.6).

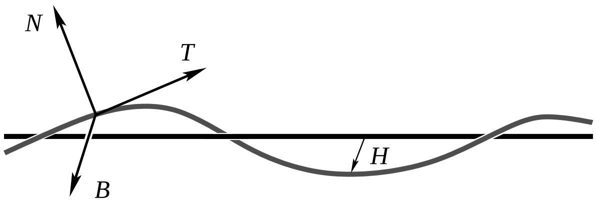

Figure 3.4: A segment of a vortex in a three-dimensional superfluid, showing the tangent vector dR/ds and two normal vectors N^1 and N^2.

Thus, for the case of a $\not{D}$-dimensional vortex in a D-dimensional flow, the source takes the form

$$J_{d_1 \cdots d_{\not{D}}}(x) = \int d^{\not{D}}s \, \varepsilon_{a_1 \cdots a_{\not{D}}} (\partial_{a_1} R_{d_1}) \cdots (\partial_{a_{\not{D}}} R_{d_{\not{D}}}) \, \delta^{(D)}(x - R(s)), \tag{3.11}$$

where summations from 1 to $\not{D}$ are implied over repeated indices a_1, a_2, etc. We show in A that this form of the source does indeed describe a generalized vortex of unit vorticity, in the sense that the circulation of the superflow computed on any one-dimensional loop that surrounds the source a single time is unity (up to a sign that depends on the sense in which the loop is traversed). Furthermore, this form of the source obeys the condition of being closed, i.e., $\partial_{d_1} J_{d_1 \cdots d_{D-2}}(x) = 0$, as we also show in A. In the language of differential forms, Eq. (3.11) is expressed as

$$J(x) = \int d^{\not{D}}s \; \delta^{(D)}(x - R(s)) \; (\partial_1 R) \wedge (\partial_2 R) \wedge \cdots \wedge (\partial_{\not{D}} R). \tag{3.12}$$

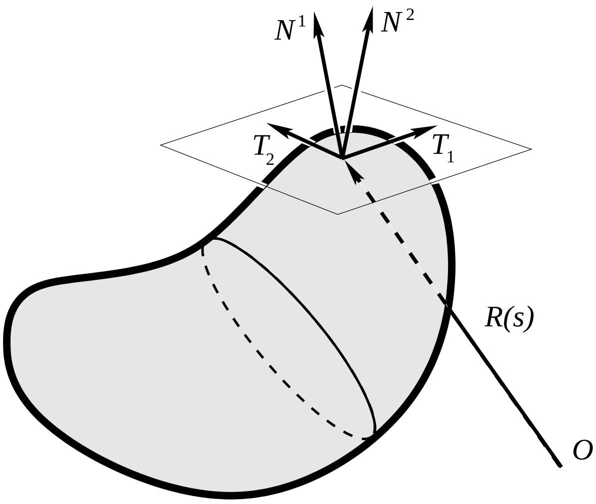

Figure 3.5: A two-dimensional vortex sub-manifold in a four-dimensional superfluid. Shown are two tangent vectors, T_1 and T_2, and two normal vectors, N^1 and N^2, at the point $R(s)$.

It is also useful to consider the dual of the source, viz., the completely skew-symmetric rank-2 field

$$\begin{aligned} \star\, J_{d_{D-1} d_D}(x) &:= \frac{1}{\not{D}!}\, \epsilon_{d_1 d_2 \cdots d_{\not{D}} d_{D-1} d_D}\, J_{d_1 \cdots d_{\not{D}}}(x) \\ &= \frac{\epsilon_{d_1 \cdots d_{D-1} d_D}}{\not{D}!} \int d^{\not{D}}s\, \varepsilon_{a_1 \cdots a_{\not{D}}} (\partial_{a_1} R_{d_1}) \cdots (\partial_{a_{\not{D}}} R_{d_{\not{D}}})\, \delta^{(D)}\,(x - R(s)). \end{aligned} \tag{3.13}$$

Let us pause to analyze the factor

$$[\not{D}!]^{-1}\, \epsilon_{d_1 \cdots d_{D-1} d_D}\, \varepsilon_{a_1 \cdots a_{\not{D}}}\, (\partial_{a_1} R_{d_1}) \cdots (\partial_{a_{\not{D}}} R_{d_{\not{D}}}) \tag{3.14}$$

from the integrand of Eq. (3.13), and hence to introduce some useful geometrical ideas. Being a skew-symmetrized (over the *spatial* indices d_1, d_2 etc.) product of the set of vectors $\{\partial_a R\}_{a=1}^{\not{D}}$ tangent to the sub-manifold at the point $R(s)$, the skew-symmetrization over indices associated with the sub-manifold coordinates (via $\varepsilon_{a_1 \cdots a_{\not{D}}}$) is elementary, yielding a factor of $\not{D}!$, so that the factor can equally well be written as

$$\epsilon_{d_1 \cdots d_D} \left(\partial_1 R_{d_1}\right) \cdots \left(\partial_{\not{D}} R_{d_{\not{D}}}\right). \tag{3.15}$$

Being orthogonal to the plane tangent to the manifold (i.e., the plane spanned by the vectors $\{\partial_a R\}_{a=1}^{\not{D}}$), this factor can be expressed in terms of a skew-symmetric product of *any pair*, N^1 and N^2, of unit vectors that are mutually orthogonal and orthogonal to the tangent plane (see Fig. 3.4 for an example in $D = 3$ and Fig. 3.5 for an example in $D = 4$), i.e., as

$$\left(N^1_{d_{D-1}} N^2_{d_D} - N^1_{d_D} N^2_{d_{D-1}}\right) \mathcal{B}, \tag{3.16}$$

in terms of a coefficient $\mathcal{B}$. To determine $\mathcal{B}$ (up to a sign associated with orientation), we take the formula

$$\left(N^1_{d_{D-1}} N^2_{d_D} - N^1_{d_D} N^2_{d_{D-1}}\right)\mathcal{B} = \epsilon_{d_1 \cdots d_D}(\partial_1 R_{d_1}) \cdots (\partial_{\not{D}} R_{d_{\not{D}}}), \tag{3.17}$$

and fully contract each side with itself to obtain

$$
\begin{aligned}
\mathcal{B}^2 &= \tfrac{1}{2}\epsilon_{d_1\cdots d_{\not D} d_{D-1} d_D}\,(\partial_1 R_{d_1})\cdots(\partial_{\not D} R_{d_{\not D}})\,\epsilon_{\bar d_1\cdots \bar d_{\not D} d_{D-1} d_D}\,(\partial_1 R_{\bar d_1})\cdots(\partial_{\not D} R_{\bar d_{\not D}}) \\
&= (\delta_{d_1\bar d_1}\delta_{d_2\bar d_2}\cdots\delta_{d_{\not D}\bar d_{\not D}} - \delta_{d_1\bar d_2}\delta_{d_2\bar d_1}\cdots\delta_{d_{\not D}\bar d_{\not D}} + \cdots) \\
&\qquad\qquad \times(\partial_1 R_{d_1})\cdots(\partial_{\not D} R_{d_{\not D}})(\partial_1 R_{\bar d_1})\cdots(\partial_{\not D} R_{\bar d_{\not D}}) \\
&= +(\partial_1 R_{d_1})(\partial_1 R_{d_1})\,(\partial_2 R_{d_2})(\partial_2 R_{d_2})\cdots(\partial_{\not D} R_{d_{\not D}})(\partial_{\not D} R_{d_{\not D}}) \\
&\quad -(\partial_1 R_{d_1})(\partial_2 R_{d_1})\,(\partial_2 R_{d_2})(\partial_1 R_{d_2})\cdots(\partial_{\not D} R_{d_{\not D}})(\partial_{\not D} R_{d_{\not D}}) + \cdots \\
&= \det g,
\end{aligned}
\tag{3.18}
$$

where $g_{a'a}$ are the components of the *induced metric* g on the vortex sub-manifold, also known as the *first fundamental form* [32], and are defined via

$$
g_{a'a}(s) := \frac{\partial R_d(s)}{\partial s_{a'}}\frac{\partial R_d(s)}{\partial s_a}. \tag{3.19}
$$

Thus, for the geometric factor (3.14) we obtain

$$
\sqrt{\det g}\,\big(N^1_{d_{D-1}}N^2_{d_D} - N^2_{d_{D-1}}N^1_{d_D}\big), \tag{3.20}
$$

and for the components of the dual source field $\star J$ we obtain

$$
\begin{aligned}
\star J_{d_{D-1}d_D}(x) &= \int d^{\not D}s\,\sqrt{\det g} \\
&\qquad \times\big(N^1_{d_{D-1}}N^2_{d_D} - N^1_{d_D}N^2_{d_{D-1}}\big)\,\delta^{(D)}(x-R(s))\,.
\end{aligned}
\tag{3.21}
$$

The measure $d^{\not D}s\,\sqrt{\det g}$ is the infinitesimal area element on the vortex sub-manifold $R(s)$. In the language of differential forms, Eq. (3.13) is expressed as

$$
\star J(x) = \int d^{\not D}s\,\sqrt{\det g}\,\big(N^1\wedge N^2\big)\,\delta^{(D)}(x-R(s))\,. \tag{3.22}
$$

3.3 Equilibrium superflows of a given vortical content

In this section, we introduce energetic considerations and show that—provided we adopt a suitable choice of gauge for the *rotational* part V of the flow—the *irrotational* contribution to the flow $\nabla\Phi$ is zero in the equilibrium state. After that, introduce a convenient gauge potential A, and use it to solve for the flow in the presence of a given vortical content, the latter being specified by a suitable source term. The gauge potential is a rank-$\not{D}$, skew-symmetric tensor, precisely analogous to the vector potential of three-dimensional magnetostatics.

3.3.1 Minimization of superflow kinetic energy

We take the energy of the system to be entirely associated with the kinetic energy of superflow. As we are addressing flows possessing a *prescribed* set of vortices, the energy associated with the absence of superfluidity in the cores of the vortices will not play a significant role. We assume that the mass-density of the superfluid ρ is constant, i.e., we take the fluid to be incompressible. The energy ρE associated with the flow $\mathcal{V}$ can therefore be specified in terms of the functional

$$E := \frac{1}{2}\int_{\Omega} d^D x\, \mathcal{V}_d(x)\, \mathcal{V}_d(x), \tag{3.23}$$

which, for convenience, we shall refer to as the energy, despite the missing factor of ρ.

To determine the equilibrium flow in the presence of some prescribed vortical content, we regard V in Eq. (3.4) as fixed and minimize E with respect to Φ. The corresponding stationarity condition reads:

$$\begin{aligned} 0 &= \frac{\delta E}{\delta\Phi(x)} = \frac{\delta}{\delta\Phi(x)}\frac{1}{2}\int_{\Omega} d^D\bar{x}\,[V_d(\bar{x}) + \nabla_d\Phi(\bar{x})]\,[V_d(\bar{x}) + \nabla_d\Phi(\bar{x})] && (3.24)\\ &= -\nabla_d\big(\nabla_d\,\Phi(x) - V_d(x)\big), && (3.25)\end{aligned}$$

where we have used the periodic boundary conditions on V and Φ to eliminate boundary terms, and thus arrive at an overall condition of divergencelessness on the flow $\mathcal{V}$. Now, V obeys Eq. (3.6), and is therefore not unique, being adjustable by any gradient of a scalar. This gauge freedom permits us to demand that the rotational part of the flow itself be divergenceless, i.e.,

$$\nabla_d V_d = 0, \tag{3.26}$$

and we shall impose this demand. Thus, from Eq. (3.25) we see that the minimizer Φ is harmonic, i.e. $\nabla_d \nabla_d \Phi = 0$, and subject to periodic boundary conditions. Thus, Φ is a constant, and does not contribute to the equilibrium flow $\mathcal{V}$, which is given by V alone, the latter obeying Eqs. (3.6) and (3.26). Thus, the flow has energy $(\rho/2) \int d^D x \, V_d V_d$. (Minimization, and not just stationarity, of the energy follows directly from the positive semi-definiteness of the second variation of E.)

3.3.2 Fixing the complete flow: equilibrium state at given vortical content

To solve Eq. (3.6) in terms of the source and, more particularly, the location of the singular surface $R(s)$, we take advantage of the divergencelessness of V, Eq. (3.26). This allows us to introduce a gauge field A, which is a completely skew-symmetric rank-$\not{D}$ tensor field, in terms of which V is given by

$$V_{d_D}(x) = \epsilon_{d_1 \cdots d_D} \, \nabla_{d_{D-1}} A_{d_1 \cdots d_{\not{D}}}(x). \tag{3.27}$$

We may subject this gauge field to a gauge condition, and the one that, for convenience, we adopt, is the transverse condition:

$$\nabla_{d_{\not{D}}} \, A_{d_1 \cdots d_{\not{D}}}(x) = 0. \tag{3.28}$$

In the language of differential forms Eqs. (3.27) and (3.28) are expressed as

$$V = \star\, dA \quad \text{and} \quad dA = 0. \tag{3.29}$$

To solve for A, we insert V, expressed in terms of A, into Eq. (3.6), to obtain

$$\epsilon_{d_1\cdots d_D}\nabla_{d_{D-1}}\epsilon_{\bar{d}_1\cdots\bar{d}_{D-1}d_D}\nabla_{\bar{d}_{D-1}}A_{\bar{d}_1\cdots\bar{d}_{\not{D}}} \;=\; 2\pi J_{d_1\cdots d_{\not{D}}}. \tag{3.30}$$

We then contract the two ϵ symbols, which yields a [$(D-1)!$ termed] sum of products of $(D-1)$ Kronecker deltas, signed with the signature of the permutation $\hat{\pi}$, i.e.,

$$\epsilon_{d_1\cdots d_D}\,\epsilon_{\bar{d}_1\cdots\bar{d}_{D-1}d_D} = \sum\nolimits_{\hat{\pi}} \mathrm{sgn}(\hat{\pi})\,\delta_{d_1\hat{\pi}(\bar{d}_1)}\cdots\delta_{d_{D-1}\hat{\pi}(\bar{d}_{D-1})}, \tag{3.31}$$

and employ the skew symmetry of A and the gauge condition (3.28) to arrive at the generalized Poisson equation, viz.,

$$\not{D}!\;\nabla_d\nabla_d\,A_{d_1\cdots d_{D-2}}(x) = 2\pi J_{d_1\cdots d_{\not{D}}}(x). \tag{3.32}$$

By virtue of the adopted gauge condition, Eq. (3.32) has the cartesian components of A uncoupled from one another. Lastly, we introduce the D-dimensional Fourier transform and its inverse, defined as

$$\begin{aligned} f(q) &= \int_\Omega d^Dx\,\mathrm{e}^{-iq\cdot x}\,f(x), && (3.33)\\ f(x) &= \int \hat{d}^Dq\,\mathrm{e}^{iq\cdot x}\,f(q), && (3.34)\end{aligned}$$

where q is a D-vector and $\hat{d}^Dq := d^Dq/(2\pi)^D$, and apply it to Eq. (3.32) to take advantage of the translational invariance of the Laplace operator. (We are implicitly taking the large-volume limit, and thus the inverse Fourier transform involves an integral over wave-vectors rather than a summation.) Thus, we arrive at an algebraic equation relating the Fourier transforms of the gauge field A and the source J, which we solve to obtain

$$A_{d_1\cdots d_{\not{D}}}(q) = -\frac{2\pi}{\not{D}!}\frac{J_{d_1\cdots d_{\not{D}}}(q)}{q\cdot q}. \tag{3.35}$$

Inverting the Fourier transform, $A(q)$, yields the real-space gauge field

$$A_{d_1\cdots d_{\not{D}}}(x) = -\frac{2\pi}{\not{D}!}\int \hat{d}^D q\,\frac{\mathrm{e}^{iq\cdot x}}{q\cdot q} J_{d_1\cdots d_{\not{D}}}(q), \tag{3.36}$$

from which, via Eq. (3.27), we obtain the Fourier representation of the real-space velocity field,

$$V_{d_D}(x) = -\frac{2\pi}{\not{D}!}\,\epsilon_{d_1\cdots d_D}\int \hat{d}^D q\,\frac{iq_{d_{D-1}}\,\mathrm{e}^{iq\cdot x}}{q\cdot q}\,J_{d_1\cdots d_{\not{D}}}(q). \tag{3.37}$$

For a source J given by a single vortex, as in Eq. (3.11), we have the following, more explicit form for the velocity field:

$$\begin{aligned} V_{d_D}(x) &= -\frac{2\pi}{\not{D}\,!}\,\epsilon_{d_1\cdots d_D}\int \hat{d}^D q\,\frac{iq_{d_{D-1}}}{q\cdot q}\mathrm{e}^{-iq\cdot x} \\ &\qquad\times\int d^{\not{D}} s\,\varepsilon_{a_1\cdots a_{D-2}}(\partial_{a_1}R_{d_1})\cdots(\partial_{a_{\not{D}}}R_{d_{\not{D}}})\,\mathrm{e}^{iq\cdot R(s)}. \end{aligned} \tag{3.38}$$

As shown in B, the q integration may be performed, yielding the result

$$\begin{aligned} V_{d_D}(x) &= \frac{1}{2\,\pi^{\not{D}/2}}\,\frac{\Gamma(\not{D}/2)}{(\not{D}-1)!}\,\epsilon_{d_1\cdots d_D} \\ &\qquad\times\int d^{\not{D}} s\,\varepsilon_{a_1\cdots a_{\not{D}}}(\partial_{a_1}R_{d_1})\cdots(\partial_{a_{\not{D}}}R_{d_{\not{D}}})\,\frac{\big(x-R(s)\big)_{d_{D-1}}}{\big|x-R(s)\big|^D}, \end{aligned} \tag{3.39}$$

in which $\Gamma(z)$ is the standard Gamma function [25].

Formula (3.39) for the velocity field V gives the *structure* of the superflow at any point x in the superfluid in terms of the shape R of a vortex. In the following sections, we shall deploy this formula to address the following issues: the velocity conferred by this flow on vortices themselves, the velocity of hyper-spherical vortices, the dispersion relation for small deformations of hyper-planar vortices, the superflow kinetic energy for vortices of arbitrary shape, and the energy-momentum relation for hyper-spherical vortices.

3.4 Dynamics of vortices in arbitrary dimensions

3.4.1 Motion of vortices: asymptotics and geometry

We have already established the flow field in the presence of a vortex, Eq. (3.38). We now examine the velocity $U(\Sigma)$ that this flow confers on points $x = R(\Sigma)$ *on* the vortex itself:

$$
\begin{aligned}
U_{d_D}(\Sigma) := V_{d_D}(x)\Big|_{x=R(\Sigma)} &= \frac{1}{2\pi^{\not{D}/2}} \frac{\Gamma\left(\not{D}/2\right)}{(\not{D}-1)!}\, \epsilon_{d_1\cdots d_D} \\
&\times \int d^{\not{D}}s\, \varepsilon_{a_1\cdots a_{\not{D}}}\, \partial_{a_1} R_{d_1}(s) \cdots \partial_{a_{\not{D}}} R_{d_{\not{D}}}(s)\, \frac{\big(R(\Sigma)-R(s)\big)_{d_{D-1}}}{\big|R(\Sigma)-R(s)\big|^{D}}.
\end{aligned}
\tag{3.40}
$$

Observe that the integral in Eq. (3.40) is logarithmically divergent; physically, it is regularized via the short-distance cut-off ξ.

We now determine the leading contribution to the velocity in a regime in which the characteristic linear dimension of the vortex and the radii of curvature of the sub-manifold supporting the vortex are large, compared with ξ. In this limit, the dominant contribution to the integral is associated with small values of the denominator. As we show in D, the asymptotic behaviour of $U(\Sigma)$ is given by

$$
\begin{aligned}
U_{d_D}(\Sigma) \approx \frac{1}{\not{D}!} \frac{\Gamma(\not{D}/2)}{2\pi^{\not{D}/2}} \frac{1}{\sqrt{\det g}}\, \mathcal{A}_{\not{D}} \ln(L/\xi)\, \epsilon_{d_1\cdots d_D}\, \varepsilon_{a_1\cdots a_{\not{D}}} \\
\times \Big\{ -\frac{1}{2} (\partial_{a_1} R_{d_1}) \cdots (\partial_{a_{\not{D}}} R_{d_{\not{D}}})\, (\partial_b \partial_{b'} R_{d_{D-1}})\, \bar{g}_{bb'} \\
- (\partial_{b'} R_{d_{D-1}})\, (\partial_{a_1} R_{d_1}) \cdots (\partial_b \partial_{a_m} R_{d_m}) \cdots (\partial_{a_{\not{D}}} R_{d_{\not{D}}})\, \bar{g}_{bb'} \\
+ \frac{1}{2} (\partial_b R_{\bar{d}})\, (\partial_{\bar{b}} \partial_{\bar{b}'} R_{\bar{d}})\, (\partial_{a_1} R_{d_1}) \cdots (\partial_{a_{\not{D}}} R_{d_{\not{D}}})\, (\partial_{b'} R_{d_{D-1}}) \\
\times \left(\bar{g}_{b\bar{b}} \bar{g}_{b'\bar{b}'} + \bar{g}_{b\bar{b}'} \bar{g}_{\bar{b}\bar{b}'} + \bar{g}_{bb'} \bar{g}_{\bar{b}\bar{b}'} \right) \Big\},
\end{aligned}
\tag{3.41}
$$

in which all terms, such as gradients of R, the metric tensor g and the inverse $\bar{g}$ of the metric tensor, are evaluated at the point Σ on the vortex sub-manifold, and the repeated indices b etc. are summed from 1 to $\not{D}$. The factor $\mathcal{A}_{\not{D}}$ is the surface area of a $\not{D}$-dimensional hyper-sphere of unit radius (see, e.g., Ref. [1]).

The asymptotic approximation to the velocity $U(\Sigma)$ of the vortex has a simple, *geometrical* structure to it. This structure becomes evident via the introduction, at each point Σ of the manifold, of a pair of vectors $N^1(\Sigma)$ and $N^2(\Sigma)$ that are normalized, orthogonal to one another, and orthogonal to the tangent plane to the vortex sub-manifold at $R(\Sigma)$, as made after Eq. (3.15). The vectors N^1 and N^2 span the manifold of dimension two complementary to vortex sub-manifold, playing the role played by the normal vector of a hyper-surface (i.e., a sub-manifold of codimension 1). The choice of the vectors $N^1(\Sigma)$ and $N^2(\Sigma)$ is not unique; alternative linear combinations of them, provided the pair remains orthonormal to one another and orthogonal to the tangent plane, will do equally well. However, what *is* independent of this choice, is unique and, from a geometric perspective, is the natural object, is the skew-symmetric product of $N^1(\Sigma)$ and $N^2(\Sigma)$, viz., the two-form

$$N := N^1 \wedge N^2, \tag{3.42}$$

the components of which are given by

$$N_{d_1 d_2} := N^1_{d_1}\, N^2_{d_2} - N^1_{d_2}\, N^2_{d_1}\,, \tag{3.43}$$

already featuring in Eq. (3.16) et seq. As we show in E, the right-hand side of Eq. (3.41) can be expressed in terms of the components of the $N^1 \wedge N^2$ two-form, which then reads

$$U_d(\Sigma) \approx \frac{\hbar}{2M}\, \bar{g}_{aa'}\, N_{dd'}\, (\partial_a \partial_{a'} R_{d'})\, \ln(L/\xi). \tag{3.44}$$

Note that we have restored to dimensional form of the velocity in Eq. (3.44), according to the prescription given in the footnote [3].

Observe the role played by the geometry here. The local shape of the vortex enters through the *intrinsic* geometric characteristic, the metric g [i.e., the first fundamental form, Eq. (3.19)], as well as through the *extrinsic* geometric characteristic $N_{dd'}\,\partial_a\partial_{a'}R_{d'}$ (i.e., the second fundamental form [32]). For the familiar case of $D = 3$, one choice for the two vectors N^1 and N^2 is the normal and the binormal to the vortex curve. Together with the tangent to the 1-dimensional vortex line they form an orthonormal basis (i.e., a Frenet-Serret frame), which reflects the extrinsic geometry of the vortex line. In contrast with its higher-dimensional extensions, the vortex line does not possess any intrinsic geometry.

3.4.2 Velocity of circular, spherical and hyper-spherical vortices: exact and asymptotic results

We now examine the velocity of vortices for the illustrative cases of circular, spherical and hyper-spherical $\not{D}$-dimensional vortices of radius L, embedded in D-dimensional space. (The case of $D = 2$ will be treated separately.) In such cases, the vortex sub-manifolds can be taken to have the following form:

$$\begin{aligned} R(s) \quad = \quad & L(\sin s_1 \sin s_2 \cdots \sin s_{\not{D}-1} \sin s_{\not{D}}, \\ & \sin s_1 \sin s_2 \cdots \sin s_{\not{D}-1} \cos s_{\not{D}}, \\ & \sin s_1 \sin s_2 \cdots \cos s_{\not{D}-1}, \\ & \ldots\ldots\ldots\ldots\ldots, \\ & \sin s_1 \cos s_2, \\ & \cos s_1, \\ & 0), \end{aligned} \tag{3.45}$$

[3]To restore the physical units, we multiply velocities by $\hbar/M$, energies by $\rho\hbar^2/M^2$, momenta by $\rho\hbar/M$ and frequencies by $\hbar/M$.

with the $\not{D}$ parameters $(s_1, \ldots, s_{\not{D}})$ respectively ranging over intervals of length $(\pi, 2\pi, \ldots, 2\pi)$, and where, for each D, we have chosen the manifold to be centered at the coordinate origin and oriented so that it is confined to the $x_D = 0$ hyper-plane. For example, in the case of $D = 3$, the vortex sub-manifold is then a ring of radius L, lying in the x_1-x_2 plane. As the vortices are circular, spherical or hyper-spherical, rotational invariance ensures that all points on them move with a common velocity, and that this velocity points along the direction perpendicular to the $(D-1)$-dimensional plane in which the vortices are confined, i.e., the D direction.

Formally, we can write an exact expression for the velocity $U(\Sigma)$ of any point Σ on the manifold, by substituting $R(s)$ into Eq. (3.40). Thus, may choose any point on the manifold at which to compute the velocity, and for convenience we choose the point given by $s_1 = 0$ [i.e., $R = (0, \ldots, 0, L, 0)$]. Two convenient vectors orthogonal to each other and normal to the vortex sub-manifold there are:

$$N^1 = (0, 0, \ldots, 0, 0, 1) \quad \text{and} \quad N^2 = (0, 0, \ldots, 0, 1, 0). \tag{3.46}$$

By substituting R, N^1 and N^2 into Eq. (3.44) we obtain

$$U_D(\Sigma) = \frac{1}{2L} \ln\left(L/\xi\right), \tag{3.47}$$

regardless of the dimension D (provided that $D > 2$).

In arriving at Eq. (3.47) it was our assumption that the vortex is a manifold. Thus, any point on the vortex has other points on the vortex in its neighbourhood; this is the origin of the logarithmic dependence on ξ. This feature does not hold for $D = 2$ because in this case the vortex is a pair of disconnected points not a manifold, and hence Eq. (3.47) does not apply. Instead, the velocity conferred on a vortex by its anti-vortex partner separated from it by a distance $2L$ is given by the well-known formula $1/2L$, and points orthogonally to the line connecting the partners.

3.4.3 Motion of weak distortions of linear, planar and hyper-planar vortices

Next, we assume that the manifold is a weak distortion of a $\not{D}$-dimensional hyperplane, and examine the motion of this distortion that results from it being transported by the flow. Such hyper-planar vortices are the generalization to the D-dimensional setting of the one-dimensional straight-line vortex in the three-dimensional setting—first studied for normal fluids by Thomson in 1880 [39]—and the flat plane vortex in the four-dimensional setting. The points on the undistorted hyper-plane are given by

$$x = F(s), \tag{3.48}$$

where $F(s) = \sum_{a=1}^{\not{D}} s_a \, e_a$, the set $\{e_d\}_{d=1}^{D}$ is a complete orthonormal Euclidian basis in the ambient space, and the variables s range without bound. To obtain a weak distortion of the flat manifold we augment $F(s)$ with a "height" function,

$$H(s) = H_{D-1}(s) \, \mathrm{e}_{D-1} + H_D(s) \, \mathrm{e}_D, \tag{3.49}$$

which has nonzero components H_{D-1} and H_D only, i.e., it points in the two directions transverse to the flat manifold. Figure 3.6 illustrates the weak distortion of a plane in $D = 4$. Thus, the points on the weak distortion of the flat manifold are given by

$$x = F(s) + H(s). \tag{3.50}$$

We now analyze the dynamics of the small perturbations (i.e., ripples) that propagate on the manifold, by constructing an equation of motion for them. The first ingredient is

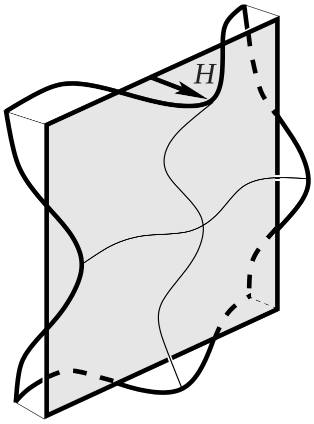

Figure 3.6: A weak distortion of a hyper-planar vortex in a four-dimensional superfluid. The height function H characterizes the distortion, which is perpendicular to the undistorted plane.

the velocity U of any point σ on the manifold, which we determine by applying Eq. (3.40), subject to the replacement

$$R(s) \to F(s) + H(s). \tag{3.51}$$

Thus, we arrive at the flow velocity at the point $x = R(\sigma)$:

$$\begin{aligned} U_{d_D}(\sigma) \quad = \quad & 2\pi \not{D} \frac{\Gamma(\not{D}/2)}{4\pi^{D/2}} \epsilon_{d_1 \cdots d_D} \int d^{\not{D}} s \, \varepsilon_{a_1 \cdots a_{\not{D}}} \\ & \times \partial_{a_1} (F_{d_1}(s) + H_{d_1}(s)) \cdots \partial_{a_{\not{D}}} (F_{d_{\not{D}}}(s) + H_{d_{\not{D}}}(s)) \\ & \times \frac{\big(F(\sigma) - F(s) + H(\sigma) - H(s)\big)_{d_{D-1}}}{\big|F(\sigma) - F(s) + H(\sigma) - H(s)\big|^D}. \end{aligned} \tag{3.52}$$

We are concerned with small distortions, and we thus expand the right hand side of Eq. (3.52) to first order in the height H; the details of this expansion are given in F. For all points σ of the manifold, the resulting velocity $U(\sigma)$ points in a direction transverse to the flat manifold, its only nonzero components being approximately given by

$$U_{\gamma'}(\sigma) \approx 2\pi \not{D} \frac{\Gamma(\not{D}/2)}{4\pi^{D/2}} \epsilon_{\gamma\gamma'} \int \frac{d^{\not{D}} s}{\left|F(\sigma) - F(s)\right|^{D}} \times \left\{ \Big(H(\sigma) - H(s)\Big)_{\gamma} - \sum_{\nu=1}^{\not{D}} (\sigma - s)_{\nu} \frac{\partial H_{\gamma}(s)}{\partial s_{\nu}} \right\}, \tag{3.53}$$

where the indices γ and γ' (which, are summed over when repeated) take only the values $(D-1)$ and D, and $\epsilon_{\gamma\gamma'}$ is the skew-symmetric tensor with $\epsilon_{D-1,D} := +1$

We pause to note that the superflow velocity field $V(x)$ is translationally covariant, in the sense that a global shift in the position of the vortex sub-manifold by a constant vector C (i.e., $R \to R + C$) produces the shift of the superflow velocity field $V(x) \to V(x - C)$. The vortex velocity given by Eq. (3.52), and its approximation Eq. (3.53), both transform in accordance with this requirement. In particular, it should be noted that Eq. (3.53) is invariant under translations of the distortion of the form $H_{\gamma}(\sigma) = H_{\gamma}(\sigma) + C_{\gamma}$.

Returning to the calculation of the vortex velocity for the case of weak distortions from hyper-planarity, the form of the denominator in the integral in Eq. (3.53) suggests that the dominant contribution to $U_{\gamma'}(\Sigma)$ comes from short distances, i.e., $s \approx \sigma$. We therefore approximate this integral by making a Taylor expansion of the term $H(\sigma)$ about $\sigma = s$ to second order, thus arriving at the approximate vortex velocity

$$U_{\gamma'}(\sigma) \approx 2\pi \not{D} \frac{\Gamma(\not{D}/2)}{4\pi^{D/2}} \epsilon_{\gamma\gamma'} \int \frac{d^{\not{D}} s}{\left(\sum_{\nu'} (s-\sigma)_{\nu'}^{2}\right)^{D/2}} \times (\sigma - s)_{\nu} (\sigma - s)_{\bar{\nu}} \frac{1}{2} \frac{\partial^2 H_{\gamma}}{\partial s_{\nu} \partial s_{\bar{\nu}}}. \tag{3.54}$$

To obtain a closed system of equations for the dynamics of the height function, we observe that the superflow carries the vortex sub-manifold with it, so that the evolution in time t of the height is given by

$$\partial H_{\gamma}(\sigma, t)/\partial t = U_{\gamma}(\sigma, t). \tag{3.55}$$

In the approximation (3.54), this equation of motion becomes a first-order differential (in time), nonlocal (in "internal space," i.e., in the coordinates σ that span the undistorted, hyper-planar vortex sub-manifold) system of two coupled linear equations that is translationally invariant in both (internal) space and time. As such, it is reducible to a two-by-two matrix algebraic form under Fourier transformation in both (internal) space and time, which we define as follows:

$$\widehat{L}(q,\omega) = \int d^{\not{D}}\sigma\, \mathrm{e}^{-iq\cdot\sigma} \int dt\, \mathrm{e}^{i\omega t}\, L(\sigma,t), \tag{3.56}$$

$$L(\sigma,t) = \int \frac{d^{\not{D}}q}{(2\pi)^{\not{D}}}\, \mathrm{e}^{iq\cdot\sigma} \int \frac{d\omega}{2\pi}\, \mathrm{e}^{-i\omega t}\, \widehat{L}(q,\omega). \tag{3.57}$$

Computing the spatial Fourier transform of Eq. (3.55) is addressed in G; the temporal Fourier transform is elementary. Introducing, for convenience, the coefficient

$$\Lambda(|k|) := \frac{\pi^{\not{D}/2}}{\Gamma(\not{D}/2)}\, k^2 \ln\left(1/|k|\xi\right), \tag{3.58}$$

arrived at via the spatial Fourier transform (see G), the equation of motion becomes

$$-i\omega \widehat{H}_\gamma(q,\omega) = \Lambda(|q|)\, \epsilon_{\gamma\gamma'}\, \widehat{H}_{\gamma'}(q,\omega). \tag{3.59}$$

The solvability condition for this system is

$$\det \begin{pmatrix} i\omega & \Lambda(|q|) \\ -\Lambda(|q|) & i\omega \end{pmatrix} = 0, \tag{3.60}$$

and gives rise to the following dispersion relation for the spectrum of normal modes of oscillation, with two modes for each value of the $\not{D}$-vector q conjugate to the position in the flat vortex sub-manifold hyper-plane:

$$\omega(|q|) = \pm\frac{\pi^{\not{D}/2}}{\Gamma(\not{D}/2)}\;\frac{\hbar}{M}\,|q|^2\ln\left(1/|q|\xi\right), \tag{3.61}$$

where we have restored the dimensional factors. The normal modes are themselves given by

$$\begin{pmatrix} \widehat{H}^{\pm}_{D-1} \\ \widehat{H}^{\pm}_{D} \end{pmatrix} \approx \begin{pmatrix} 1 \\ \mp i \end{pmatrix} e^{i(q\sigma\mp\omega t)}, \tag{3.62}$$

and these correspond to left and right circularly polarized oscillations.

3.5 Superflow kinetic energy for given vortical content

3.5.1 Exact evaluation

We now use the solution for the velocity field, Eq. (3.37), together with the equation for the kinetic energy E of the velocity field, Eq. (3.23), to obtain the kinetic energy in terms of the source J:

$$\begin{aligned}
E &= \frac{1}{2}\int_\Omega d^Dx\, V_d(x)\, V_d(x) = \frac{1}{2}\int_\Omega d^Dx\, V_d^*(x)\, V_d\,(x) \\
&= \frac{1}{2}\delta_{d'_D d_D}\left(\frac{2\pi}{\not{D}!}\right)^2 \int_\Omega d^Dx\, \epsilon_{d'_1\cdots d'_D}\int \hat{d}^Dq'\, q'_{d'_{D-1}}\, \mathrm{e}^{-iq'\cdot x}\, \frac{J^*_{d'_1\cdots d'_{\not{D}}}(q')}{q'\cdot q'} \\
&\qquad\qquad \times \epsilon_{d_1\cdots d_D}\int \hat{d}^Dq\, q_{d_{D-1}}\, \mathrm{e}^{iq\cdot x}\, \frac{J_{d_1\cdots d_{\not{D}}}(q)}{q\cdot q} \\
&= \frac{1}{2}\delta_{d'_D d_D}\left(\frac{2\pi}{\not{D}!}\right)^2 \epsilon_{d'_1\cdots d'_D}\int \hat{d}^Dq'\, q'_{d'_{D-1}}\, \frac{J^*_{d'_1\cdots d'_{\not{D}}}(q')}{q'\cdot q'} \\
&\qquad\qquad \times \epsilon_{d_1\cdots d_D}\int \hat{d}^Dq\, q_{d_{D-1}}\, \frac{J_{d_1\cdots d_{\not{D}}}(q)}{q\cdot q}\, \hat{\delta}^{(D)}(q'-q) \\
&= \frac{1}{2}\delta_{d'_D d_D}\left(\frac{2\pi}{\not{D}!}\right)^2 \epsilon_{d'_1\cdots d'_D}\epsilon_{d_1\cdots d_D} \\
&\qquad\qquad \times \int \hat{d}^Dq\, J^*_{d'_1\cdots d'_{\not{D}}}(q)\, J_{d_1\cdots d_{\not{D}}}(q)\, \frac{q_{d'_{D-1}}\, q_{d_{D-1}}}{(q\cdot q)^2},
\end{aligned} \tag{3.63}$$

where $\hat{\delta}^{(D)}(q) := (2\pi)^D\delta^{(D)}(q)$. For a source J given by a single vortex, as in Eq. (3.11), we then have the more explicit form for the energy:

$$\begin{aligned}
E = \frac{\delta_{d'_D d_D}}{2}\left(\frac{2\pi}{\not{D}!}\right)^2 &\epsilon_{d'_1\cdots d'_D}\epsilon_{d_1\cdots d_D}\int d^{\not{D}}s'\, \varepsilon_{a'_1\cdots a'_{\not{D}}}\big(\partial_{a'_1}R_{d'_1}(s')\big)\cdots\big(\partial_{a'_{\not{D}}}R_{d'_{\not{D}}}(s')\big) \\
&\times \int d^{\not{D}}s\, \varepsilon_{a_1\cdots a_{\not{D}}}\big(\partial_{a_1}R_{d_1}(s)\big)\cdots\big(\partial_{a_{\not{D}}}R_{d_{\not{D}}}(s)\big) \\
&\qquad \times \int \hat{d}^Dq\, \mathrm{e}^{iq\cdot(R(s')-R(s))}\frac{q_{d'_{D-1}}\, q_{d_{D-1}}}{(q\cdot q)^2}.
\end{aligned} \tag{3.64}$$

Writing X for $(R(s') - R(s))$ and using the result for the q integration given in C, the energy becomes

$$\begin{aligned}
E = \frac{1}{2}\left(\frac{2\pi}{\not{D}!}\right)^2 &\epsilon_{d'_1\cdots d'_D}\epsilon_{d_1\cdots d_D}\delta_{d'_D d_D}\int d^{\not{D}}s'\, \varepsilon_{a'_1\cdots a'_{\not{D}}}\big(\partial_{a'_1}R_{d'_1}(s')\big)\cdots\big(\partial_{a'_{\not{D}}}R_{d'_{\not{D}}}(s')\big) \\
&\times \int d^{\not{D}}s\, \varepsilon_{a_1\cdots a_{\not{D}}}\big(\partial_{a_1}R_{d_1}(s)\big)\cdots\big(\partial_{a_{\not{D}}}R_{d_{\not{D}}}(s)\big) \\
&\qquad \times \frac{1}{(X\cdot X)^{\not{D}/2}}\left(\delta_{d'_{D-1}d_{D-1}}\,\mathcal{P}_1 + \frac{X_{d'_{D-1}}\, X_{d_{D-1}}}{X\cdot X}\,\mathcal{P}_2\right),
\end{aligned} \tag{3.65}$$

where $\mathcal{P}_1$ and $\mathcal{P}_2$ depend only on D and are given in C.

The final step in constructing the energy of a single vortex involves contracting the D-dimensional ϵ symbols and taking advantage of the skew-symmetry of the sources. Thus, we arrive at the result

$$
\begin{aligned}
E = \frac{1}{2}\left(\frac{2\pi}{\not{D}!}\right)^2 \frac{1}{(2\pi)^D} \mathcal{A}_{D-1}\sqrt{\pi}\,\Gamma\left(\tfrac{D}{2}-\tfrac{1}{2}\right)\, 2^{\not{D}-2}\,\Gamma\left(\tfrac{D}{2}-1\right) \\
\times \int d^{\not{D}} s'\, \varepsilon_{a'_1 \cdots a'_{\not{D}}} \big(\partial_{a'_1} R_{d_1}(s')\big) \cdots \big(\partial_{a'_{\not{D}-1}} R_{d_{\not{D}-1}}(s')\big)\big(\partial_{a'_{\not{D}}} R_{d'_{\not{D}}}(s')\big) \\
\times \int d^{\not{D}} s\, \varepsilon_{a_1 \cdots a_{\not{D}}} \big(\partial_{a_1} R_{d_1}(s)\big) \cdots \big(\partial_{a_{\not{D}-1}} R_{d_{\not{D}-1}}(s)\big)\big(\partial_{a_{\not{D}}} R_{d_{\not{D}}}(s)\big) \\
\times \frac{1}{(X\cdot X)^{\not{D}/2}} \Big\{ (4-D)\not{D}!\, \delta_{d'_{\not{D}} d_{\not{D}}} + \not{D}^2 \not{D}!\, \frac{X_{d'_{\not{D}}} X_{d_{\not{D}}}}{X\cdot X} \Big\}.
\end{aligned}
\tag{3.66}
$$

The factor $\mathcal{A}_{D-1}$ is the surface area of a $(D-1)$-dimensional hyper-sphere of unit radius (see, e.g., Ref. [1]), and $\Gamma(z)$ is the standard Gamma function (see, e.g., Ref. [25]).

Formula (3.66) gives the kinetic energy in terms of the local geometry of "patches" of the vortex sub-manifold and interactions between such "patches." The asymptotically dominant contributions to the energy come from nearby patches, and we evaluate these contributions in the following section. The linearity of the theory ensures that the energy associated with configurations involving multiple, rather than just one, vortices is obtained via an elementary extension, comprising self-interactions terms, such as the one just given, for each vortex, as well as mutual interaction terms for pairs of distinct vortices.

3.5.2 Asymptotics and geometry

We now analyze the energy of a vortex asymptotically, subject to the conditions discussed earlier in Section 3.4.1, (i.e., the characteristic linear dimensions and radii of curvature of the vortex are much larger than the short-distance cut-off ξ). In this regime, the dominant contribution to the energy is associated with the large values of the integrand that occur when the ($\not{D}$-dimensional) integration variables s and s' are nearly coincident, so that the denominator $(X\cdot X)^{\not{D}/2}$ is small. To collect such contributions, it is useful to change variables from (s,s') to (Σ,σ) as follows:

$$s_a \;:=\; \Sigma_a, \tag{3.67}$$

$$s'_a \;:=\; \Sigma_a+\sigma_a, \tag{3.68}$$

the Jacobian for this transformation being unity,

$$\frac{\partial(s,s')}{\partial(\Sigma,\sigma)} = \left[\det\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\right]^{\not{D}} = 1, \tag{3.69}$$

so that $\int d^{\not{D}}s' \int d^{\not{D}}s \ldots = \int d^{\not{D}}\Sigma \int d^{\not{D}}\sigma \ldots$.

The energy formula (3.66) comprises two pieces, and we now consider them in turn. The first, (a), involves the integral

$$\int d^{\not{D}}\Sigma \int \frac{d^{\not{D}}\sigma}{(X\cdot X)^{\not{D}/2}}\, \varepsilon_{a'_1\cdots a'_{\not{D}}} \big(\partial_{a'_1} R_{d_1}(\Sigma-s)\big)\cdots\big(\partial_{a'_{\not{D}}} R_{d_{\not{D}}}(\Sigma-s)\big) \\ \times\varepsilon_{a_1\cdots a_{\not{D}}}\big(\partial_{a_1}R_{d_1}(s)\big)\cdots\big(\partial_{a_{\not{D}}}R_{d_{\not{D}}}(s)\big). \tag{3.70}$$

The second, (b), involves the integral

$$\int d^{\not{D}}\Sigma \int d^{\not{D}}\sigma\, \frac{X_{d'_{\not{D}}}X_{d_{\not{D}}}}{(X\cdot X)^{D/2}} \\ \times\varepsilon_{a'_1\cdots a'_{\not{D}}}\big(\partial_{a'_1}R_{d_1}(\Sigma-s)\big)\cdots\big(\partial_{a'_{\not{D}-1}}R_{d_{\not{D}-1}}(\Sigma-s)\big)\big(\partial_{a'_{\not{D}}}R_{d'_{\not{D}}}(\Sigma-s)\big) \\ \times\varepsilon_{a_1\cdots a_{\not{D}}}\big(\partial_{a_1}R_{d_1}(s)\big)\cdots\big(\partial_{a_{\not{D}-1}}R_{d_{\not{D}-1}}(s)\big)\big(\partial_{a_{\not{D}}}R_{d_{\not{D}}}(s)\big). \tag{3.71}$$

Our strategy will be to Taylor-expand the integrands in terms (a) and (b) about the point $\sigma=0$, keeping only the leading-order terms.

Let us focus on term (a). It is adequate to approximate the numerator at zeroth order:

$$\varepsilon_{a'_1\cdots a'_{\not{D}}}\big(\partial_{a'_1}R_{d_1}(\Sigma)\big)\cdots\big(\partial_{a'_{\not{D}}}R_{d_{\not{D}}}(\Sigma)\big)\,\varepsilon_{a_1\cdots a_{\not{D}}}\,\big(\partial_{a_1}R_{d_1}(\Sigma)\big)\cdots\big(\partial_{a_{\not{D}}}R_{d_{\not{D}}}(\Sigma)\big)\,.$$

Note the occurrence of $\not{D}$ factors of the induced metric $g_{aa'}$, defined in Eq. (3.19). In terms of g, the numerator of term (a) becomes

$$\varepsilon_{a'_1\cdots a'_{\not{D}}}\,\varepsilon_{a_1\cdots a_{\not{D}}}\,g_{a'_1a_1}(\Sigma)\,g_{a'_2a_2}(\Sigma)\cdots g_{a'_{\not{D}}a_{\not{D}}}(\Sigma), \tag{3.72}$$

and, by the well-known formula for determinants, this is

$$\not{D}!\,\det \mathrm{g}\,. \tag{3.73}$$

To complete the approximate computation of term (a) we examine the denominator,

$$(X\cdot X)^{-\not{D}/2} = |R(\Sigma+\sigma)-R(\Sigma)|^{-\not{D}}. \tag{3.74}$$

By Taylor-expanding for small σ and retaining only the leading-order contribution, we find that the denominator becomes

$$(\sigma_{a'}\,g_{a'a}\,\sigma_a)^{-\not{D}/2}\,. \tag{3.75}$$

Thus, for term (a) we have the approximate result

$$\int d^{\not{D}}\Sigma\int d^{\not{D}}\sigma\,\frac{\not{D}!\,\det g(\Sigma)}{(\sigma_{a'}\,g_{a'a}(\Sigma)\,\sigma_a)^{\not{D}/2}}\,. \tag{3.76}$$

A convenient way to analyze the integration over σ is to make a $\not{D}$-dimensional orthogonal transformation from σ to coordinates ζ that diagonalize g, the eigenvalues of which we denote by $\{\gamma_a\}_{a=1}^{\not{D}}$, and then to "squash" these coordinates to $\hat{\zeta}$ (defined via $\hat{\zeta}_a := \sqrt{\gamma_a}\,\zeta_a$), so as to "isotropize" the quadratic form in the denominator. Following these steps, postponing to the following paragraph a discussion of the integration limits, and omitting an overall factor of $\not{D}!$, our approximation for term (a) becomes

$$
\begin{aligned}
&\int d^{\not{D}}\Sigma\;\det g(\Sigma)\int \frac{d^{\not{D}}\sigma}{\left(\sigma_{a'}\,g_{a'a}(\Sigma)\,\sigma_a\right)^{\not{D}/2}} \\
&\qquad = \int d^{\not{D}}\Sigma\;\det g(\Sigma)\int \frac{d^{\not{D}}\zeta}{\left[\sum_{a=1}^{\not{D}}\gamma_a(\Sigma)\,\zeta_a^2\right]^{\not{D}/2}} \\
&\qquad = \int d^{\not{D}}\Sigma\;\frac{\det g(\Sigma)}{\prod_{a=1}^{\not{D}}\sqrt{\gamma_a(\Sigma)}}\int \frac{d^{\not{D}}\hat{\zeta}}{\left[\sum_{a=1}^{\not{D}}\hat{\zeta}_a^2\right]^{\not{D}/2}} \\
&\qquad = \int d^{\not{D}}\Sigma\;\sqrt{\det g(\Sigma)}\,\mathcal{A}_{\not{D}}\int \frac{dZ\,Z^{\not{D}-1}}{Z^{\not{D}}},
\end{aligned}
$$

where $Z := \sqrt{\sum_{a=1}^{\not{D}}\hat{\zeta}_a^2}$.

Up to now, we have been vague about the limits on the integrations over the σ variables. In the short-distance regime there is, for each value of Σ, a physically motivated cutoff that eliminates contributions from regions of σ in which the separation $|R(\Sigma+\sigma)-R(\Sigma)|$ is of order ξ or smaller. This cutoff is associated with the fact that our overall description is not intended (and is indeed unable) to capture physics on length-scales associated with the linear dimension ξ of the core of the vortices. In the long-distance regime, there are integration limits associated with the finite size of the vortex, which we take to be of order L. (Thus, we rule out of consideration vortices whose shape is strongly anisotropic; already ruled out are, e.g., rough vortices, for which the Taylor expansion of the functions describing the shapes of the vortices would be unjustified.) In particular, the last step in our approximate computation of term (a), in which we transformed to the single radial coordinate in the $\not{D}$-dimensional flat plane tangent to the vortex at the point $s=\Sigma$, is only valid if the limits of integration possess the rotational symmetry of this tangent plane, which they typically do not. (The case of $D=3$ is an exception.) However, to within the logarithmic accuracy to which we are working, it is adequate to ignore these complications and approximate what has become the Z integral by $\ln(L/\xi)$. Any refinement would just change the argument of the logarithm by a multiplicative numerical factor, and this would produce an additive correction to our result that is of sub-leading order. Thus, our approximation to term (a) becomes

$$\not{D}! \int d^{\not{D}}\Sigma \sqrt{\det \mathrm{g}(\Sigma)}\, \mathcal{A}_{\not{D}} \ln(L/\xi). \tag{3.77}$$

The remaining integral, $\int d^{\not{D}}\Sigma \sqrt{\det \mathrm{g}(\Sigma)}$, is familiar from differential geometry, and simply gives the $\not{D}$-dimensional volume of the sub-manifold supporting the vortex, which we shall denote by $\mathcal{W}_{\not{D}}$ [and which is a functional of $R(\cdot)$]. Thus, our approximation to term (a) becomes

$$\not{D}!\, \mathcal{A}_{\not{D}}\, \mathcal{W}_{\not{D}}\, \ln(L/\xi). \tag{3.78}$$

A similar procedure allows us to find an adequate approximation to term (b), as we now show. In contrast with term (a), owing to the two factors of $X_d := R_d(\Sigma+\sigma) - R_d(\Sigma)$, the numerator vanishes at zeroth order in σ, and requires Taylor expansion,

$$X_d \approx \sigma_a\, \partial R_d / \partial \Sigma_a, \tag{3.79}$$

to identify the leading-order contribution to the numerator. The remaining factors in the numerator can be evaluated at zeroth order, which then becomes

$$\begin{aligned}
&\varepsilon_{a'_1 \cdots a'_{\not{D}}}\, \partial_{a'_1} R_{d_1} \cdots \partial_{a'_{\not{D}-1}} R_{d_{\not{D}-1}}\, \partial_{a'_{\not{D}}} R_{d'_{\not{D}}}\, \varepsilon_{a_1 \cdots a_{\not{D}}}\, \partial_{a_1} R_{d_1} \cdots \partial_{a_{\not{D}-1}} R_{d_{\not{D}-1}}\, \partial_{a_{\not{D}}} R_{d_{\not{D}}} \\
&\qquad \times \sigma_{a'_{D-1}}\, \partial_{a'_{D-1}} R_{d'_{\not{D}}}\, \sigma_{a_{D-1}}\, \partial_{a_{D-1}} R_{d_{\not{D}}} \\
&= \varepsilon_{a_1 \cdots a_{\not{D}}}\, \varepsilon_{a'_1 \cdots a'_{\not{D}}}\, g_{a'_1 a_1} \cdots g_{a'_{D-3} a_{D-3}}\, g_{a'_{\not{D}} a'_{D-1}}\, \sigma_{a'_{D-1}}\, g_{a_{\not{D}} a_{D-1}}\, \sigma_{a_{D-1}} \\
&= \varepsilon_{a_1 \cdots a_{\not{D}-1} a_{\not{D}}}\, \varepsilon_{a_1 \cdots a_{\not{D}-1} a'_{D-1}}\, \sigma_{a'_{D-1}}\, g_{a_{\not{D}} a_{D-1}}\, \sigma_{a_{D-1}} \\
&= (\not{D}-1)!\, \delta_{a_{\not{D}} a'_{D-1}}\, \sigma_{a'_{D-1}}\, g_{a_{\not{D}} a_{D-1}}\, \sigma_{a_{D-1}} = \not{D}!\ \sigma_{a'}\, g_{a' a}\, \sigma_a\,,
\end{aligned} \tag{3.80}$$

in which all terms involving R and g are evaluated at the point Σ, and where we have made use of the formula for the determinant,

$$\varepsilon_{a'_1\cdots a'_{\not D}}\, g_{a'_1 a_1}\, g_{a'_2 a_2}\cdots g_{a'_{\not D} a_{\not D}} = \varepsilon_{a_1\cdots a_{\not D}} \det g, \tag{3.81}$$

and, once again, we introduced the induced metric $g_{a'a}(\Sigma)$ as well as the contraction of the skew symmetric symbols, $\varepsilon_{a_1\cdots a_{\not D-1}a}\,\varepsilon_{a_1\cdots a_{\not D-1}a'} = (\not D-1)!\,\delta_{aa'}$. As for the the denominator of term (b), this we Taylor-expand in the manner already used for term (a), noting that the overall powers in these denominators differ by unity. Thus, apart from an overall factor of $(\not D-1)!$, we have for term (b) the approximate result

$$\begin{aligned}\int d^{\not D}\Sigma \det g(\Sigma) \int d^{\not D}\sigma\, &\frac{\sigma_{a'}\, g_{a'a}(\Sigma)\,\sigma_a}{\left(\sigma_{a'} g_{a'a}(\Sigma)\,\sigma_a\right)^{D/2}} \\ &= \int d^{\not D}\Sigma \det g(\Sigma) \int \frac{d^{\not D}\sigma}{\left(\sigma_{a'}\, g_{a'a}(\Sigma)\,\sigma_a\right)^{\not D/2}},\end{aligned} \tag{3.82}$$

which is of precisely the structure that arose in the analysis of term (a), and thus the approximate result for term (b) becomes

$$(\not D-1)!\,\mathcal{A}_{\not D}\,\mathcal{W}_{\not D}\,\ln(L/\xi). \tag{3.83}$$

Finally, we assemble terms (a) and (b) to arrive at an asymptotic formula for the energy of the equilibrium flow in the presence of a vortical defect, which reads

$$E \approx \frac{\pi^{2-D}}{4}\sqrt{\pi}\,\Gamma\left(\tfrac{D}{2}-\tfrac{1}{2}\right)\Gamma\left(\tfrac{D}{2}-1\right)\mathcal{A}_{D-1}\,\mathcal{A}_{\not D}\,\ln(L/\xi)\,\mathcal{W}_{\not D}. \tag{3.84}$$

This formula simplifies, using the equation for $\mathcal{A}_\Delta$, Ref. [1]), to give

$$E \approx \left(\rho\frac{\hbar^2}{M^2}\right)\pi\,\mathcal{W}_{\not D}\,\ln(L/\xi), \tag{3.85}$$

where we have restored the dimensional factors.

3.5.3 Elementary argument for the energy

In the previous subsection, 3.5.2, we have shown that, in the asymptotic limit of a large, smooth, $\not{D}$-dimensional vortex sub-manifold having the topology of the surface of a $(D-1)$-dimensional hyper-sphere, the kinetic energy depends on the geometry of the sub-manifold only through its $\not{D}$-dimensional volume $\mathcal{W}_{\not{D}}$. This result is consistent with the following elementary argument.

Imagine starting with such a vortex sub-manifold, and "morphing" its shape into that of a hyper-sphere having the same $\not{D}$-dimensional volume. Next, cut the hyper-sphere along an equator, so that it becomes two hyper-hemispheres of equal size. Then, distort the hyper-hemispheres into flat hyper-planes separated by αL (i.e., a distance of order L), maintaining their $\not{D}$-dimensional volumes at $\frac{1}{2}\mathcal{W}_{\not{D}}$, and imposing periodic boundary conditions on opposing sides of the hyper-planes. This last step is bound to involve some stretching of the sub-manifolds, but let us assume that this will only affect the energy sub-dominantly. We choose the separation of the hyper-planes to be $\mathcal{O}(L)$, so as to best maintain the geometry of the original sub-manifold. We can think of the resulting D-dimensional system as comprising a $\not{D}$-dimensional stack of identical, two-dimensional, films of superfluid having thicknesses ξ in every one of their $\not{D}$ thin dimensions, each film containing a point-like vortex–anti-vortex pair.

Now, owing to the translational and inversion symmetry of the stack, the flow in each film is confined to the film and is simply that associated with vortices in the film, and therefore the energy of the flow is given by the formula: $2\pi\xi^{\not{D}}\,\ln(\alpha L/\xi)$ (see, e.g., Ref. [40]). Furthermore, the number of films is $\frac{1}{2}\mathcal{W}_{\not{D}}/\xi^{\not{D}}$. Thus, consistent with Eq. (3.85), the total energy is given by

$$\frac{\mathcal{W}_{\not{D}}}{2\xi^{\not{D}}} \times 2\pi\xi^{\not{D}}\,\ln\left(\frac{\alpha L}{\xi}\right) = \pi\mathcal{W}_{\not{D}}\left\{\ln\frac{L}{\xi} + \mathcal{O}(1)\right\}, \tag{3.86}$$

where the $\mathcal{O}(1)$ correction is associated with the factor α.

Sub-leading corrections to this result for the energy will not depend solely on the vortex sub-manifold volume but also on its more refined local and global geometrical characteristics.

3.5.4 Energy of circular, spherical and hyper-spherical vortices: exact and asymptotic results

In this section, we analyze the energy of a geometrically (and not just topologically) hyper-spherical vortex sub-manifold. The function $R(s)$ for such a vortex is given by Eq. (3.45), and by substituting this form into Eq. (3.65) we obtain a formally exact result for the energy.

To illustrate this point, consider the case of a two-dimensional vortex in $D = 4$, for which the energy reads

$$\begin{aligned} E = \frac{1}{8} L^2 \int_{-\pi}^{\pi} ds_1\, ds_1' \int_{-\pi/2}^{\pi/2} ds_2\, ds_2' \, \cos(s_2)\, \cos(s_2') \\ \times \left\{ \frac{2}{1 - \cos(s_1 - s_1')\, \cos s_2\, \cos s_2' - \sin s_2\, \sin s_2'} - 1 \right\}. \end{aligned} \tag{3.87}$$

As with the case of arbitrary D, the integrand in Eq. (3.87) is singular for $(s_1, s_2) = (s_1', s_2')$; therefore, we follow the procedure, described in detail in Section 3.5.2, of expanding the integrand around this point of singularity, hence obtaining the asymptotic result that the energy is given by

$$E \approx 4\pi^2\, L^2\, \ln(L/\xi). \tag{3.88}$$

Turning now to arbitrary dimension D, the energy of a hyper-spherical vortex can be obtained by replacing the volume of the manifold in Eq. (3.85) by the surface area of a $(D-1)$-dimensional hyper-sphere, viz., $\mathcal{A}_{D-1} L^{\not{D}}$ (see Ref. [1]), so that we arrive at

$$E \approx \pi\, \mathcal{A}_{D-1}\, L^{\not{D}}\, \ln(L/\xi). \tag{3.89}$$

3.6 Energy-momentum relation for vortices

We now use the results for the energies and velocities of circular, spherical and hyper-spherical vortices (i.e., $D \geq 3$), obtained respectively in Secs. 3.4.2 and 3.5.4, to construct the momentum P of such vortices and, hence, their energy-momentum relations. To do this, we observe that the energy $E(L)$ and velocity $V(L)$ depend parametrically on the radius of the vortex L. Hence, we may construct the parametric dependence of the momentum $P(L)$ via the Hamiltonian equation

$$V(L) = \frac{\partial}{\partial p} E(p) \bigg|_{p=P(L)} = \frac{\partial E/\partial L}{\partial P/\partial L}, \tag{3.90}$$

in view of which we have

$$\frac{\partial P}{\partial L} = \frac{\partial E/\partial L}{V(L)} \approx 2\pi \, \not{D} \, \mathcal{A}_{D-1} \, L^{\not{D}}, \tag{3.91}$$

where we have retained only the leading behavior at large L. By integrating with respect to L we obtain the parametric dependence of the momentum on the vortex radius,

$$P(L) \approx 2\pi \frac{D-2}{D-1} \mathcal{A}_{D-1} \, L^{D-1}, \tag{3.92}$$

and by eliminating L in favour of E we arrive at the energy-momentum relation

$$E \approx \eta(D) \, P^{\frac{D-2}{D-1}} \left(\ln \frac{P\xi}{\hbar} + \ln \frac{M}{\rho \xi^D} \right), \tag{3.93}$$

where $\eta(D)$ is a D-dependent constant given by

$$\eta(D) = \frac{(\pi \mathcal{A}_{D-1})^{1/(D-1)}}{D-1} \left(\frac{D-1}{2D-4} \right)^{\frac{D-2}{D-1}}, \tag{3.94}$$

and the sub-leading logarithmic term is sensitive to the number of particles per coherence volume.

For the case of $D = 2$, the foregoing derivation of the dispersion relation does not hold because (i) the computation of $\partial E/\partial L$ misses the leading term, which is a logarithm in L [see Eq. (3.85)]; and (ii) the formula used for $V(L)$ does not hold in $D = 2$, as noted at the end of Sec. 3.4.2. Nevertheless, if we substitute the correct forms for $\partial E/\partial L$ and $V(L)$, viz., $2\pi/L$ and $1/2L$ respectively, into Eq. (3.90), we obtain a dispersion relation identical to the $D \to 2$ limit of that given in Eq.(3.93), up to a numerical factor in the argument of the logarithm. It is amusing to note that even the amplitude $\eta(D)$ has the correct value, 2π, in the limit $D \to +2$.

3.7 Scaling and dimensional analysis

To ease the presentation, we have chosen to drop factors of $\hbar$, M and ρ from our derivations, restoring physical units only when giving results at the ends of the various sections. It is worth noting, however, that many of our results can be obtained via scaling and dimensional analysis, up to overall dimension-dependent factors and cutoff-dependent logarithms, as we now discuss.

There are two characteristic length-scales: the linear dimension of a vortex L, and the vortex core size ξ. The latter serves as a cutoff, and therefore only features in logarithmic factors, which are inaccessible to our scaling and dimensional analysis. There are two characteristic mass-scales: the mass of each of the condensing particles M, and ρL^D, where ρ is the mass-density of the superfluid. Lastly, there is a characteristic frequency-scale: $\hbar/ML^2$.

To ascertain the scaling form of the velocity of a vortex, we note that its derivation involves only the generalized D-dimensional Ampère-Maxwell law, and thus is insensitive to

ρ. Hence, dimensional analysis suggests that

$$V \sim L \times \frac{\hbar}{ML^2} \sim \frac{\hbar}{ML}, \tag{3.95}$$

which agrees, e.g., with Eq. (3.47), except for numerical and logarithmic factors. As for the energy of a flow associated with a vortex, it is proportional to the density ρ, and therefore dimensional analysis suggests that

$$E \sim \rho L^D \times L^2 \times \left(\frac{\hbar}{ML^2}\right)^2 \sim \rho L^{\not{D}} \frac{\hbar^2}{M^2}, \tag{3.96}$$

which agrees, e.g., with Eq. (3.85), except for numerical and logarithmic factors. Regarding the momentum, it was computed from the Hamiltonian equation of motion (3.90), and thus, like the energy, scales as ρ^1. Dimensional analysis therefore suggests that

$$P \sim \rho L^D \times L \times \frac{\hbar}{ML^2} \sim \rho L^{D-1} \frac{\hbar}{M}, \tag{3.97}$$

which agrees, e.g., with Eq. (3.92), except for numerical and logarithmic factors. Concerning the frequency of oscillations of distortions of wave-vector q of a hyper-planar vortex [see Eq. (3.61)], these are driven by the flow velocity, and therefore do not depend on ρ. Moreover, the undistorted vortex does not have a system-size-independent characteristic size; instead, the characteristic length-scale is set by q^{-1}. These scaling notions, together with dimensional analysis, suggest that

$$\omega \sim \frac{\hbar}{M} q^2, \tag{3.98}$$

consistent with Eq. (3.61), except for numerical and logarithmic factors.

Chapter 4

Superfluid transition in arbitrary dimensions

In this chapter we investigate the normal-to-superfluid transition in an arbitrary number of dimensions. In Chapter 2 we gave a brief exposure of this transition for the two-dimensional XY model and helium films. The three-dimensional case has been well studied; for an account of the transition of the XY three-dimensional model see e.g. [36, 5, 4, 28]; an account of a vortex ring approach to the normal-to-superfluid transition (i.e., λ-transition) is discussed in [41, 42].

4.1 Collection of vortices in a D-dimensional superfluid

When a superfluid that fills a D-dimensional volume Ω we expect more than one vortex present in the body of the fluid. The purpose of this section is to analyze the interaction between such vortices, as well as their self-energies. As in the previous chapter, we define the Hamiltonian to be a kinetic energy of the superflow:

$$\mathcal{H} = \frac{1}{2}K \int_{\Omega} d^D x \, V^*_{d_D}(x) \, V_{d_D}(x), \tag{4.1}$$

where K is the analogue of the D-dimensional superfluid stiffness defined for the two-dimensional case in Eq.(2.8), i.e.,

$$K = \frac{\rho}{k_B T}, \tag{4.2}$$

where k_B is Boltzman constant and T is the temperature. Let us assume that in the volume Ω there are i vortices, each specified by the sub-manifold $x = R^{(i)}(s)$, and each giving rise to a "current" $J^{(i)}$. The differential Ampère-Maxwell equation for the velocity will then have a superposition of sources on the right hand side:

$$\epsilon_{d_1 \cdots d_D} \nabla_{d_{D-1}} V_{d_D}(x) = 2\pi \sum_i J^{(i)}_{d_1 \cdots d_{\not{D}}}(x). \tag{4.3}$$

As for the case of a single vortex, we introduce a gauge potential A, subject to the transverse gauge condition $\nabla_{d_{\not{D}}} A_{d_1 \cdots d_{\not{D}}}(x) = 0$, such that

$$V_{d_D}(x) = \epsilon_{d_1 \cdots d_D} \nabla_{d_{D-1}} A_{d_1 \cdots d_{\not{D}}}(x). \tag{4.4}$$

to solve for A we combine Eqs. (4.3) and (4.4) thus obtaining

$$\begin{aligned} &\epsilon_{d_1 \cdots d_D} \nabla_{d_{D-1}} \epsilon_{\bar{d}_1 \cdots \bar{d}_{D-1} d_D} \nabla_{\bar{d}_{D-1}} A_{\bar{d}_1 \cdots \bar{d}_{\not{D}}} \\ &\qquad = -\not{D}! \nabla_d \nabla_d A_{d_1 \cdots d_{\not{D}}} = 2\pi \sum_i J^{(i)}_{d_1 \cdots d_{\not{D}}}. \end{aligned} \tag{4.5}$$

In the last step we used the well-known formula for contracting two skew symmetric tensors

$$\epsilon_{d_1 \cdots d_D}\, \epsilon_{\bar{d}_1 \cdots \bar{d}_{D-1} d_D} = \sum\nolimits_{\hat{\pi}} \operatorname{sgn}(\hat{\pi})\, \delta_{d_1 \hat{\pi}(\bar{d}_1)} \cdots \delta_{d_{D-1} \hat{\pi}(\bar{d}_{D-1})},$$

where $\operatorname{sgn}(\hat{\pi})$ is the signature of the permutation $\hat{\pi}(d_1 \cdots d_D)$ of D indices. Finally we introduce the D-dimensional Fourier transform and its inverse

$$f(q) = \int_\Omega d^D x\, \mathrm{e}^{-iq\cdot x}\, f(x), \tag{4.6}$$

$$f(x) = \int \hat{d}^D q\, \mathrm{e}^{iq\cdot x}\, f(q), \tag{4.7}$$

where q is a D-vector and $\hat{d}^D q := d^D q/(2\pi)^D$. We implicitly assume that the position integral is over a large volume Ω, hence the the inverse Fourier transform is an integral over wave vectors rather than a summation. In terms of the Fourier transforms, Eq. (4.5) reads:

$$A_{d_1\cdots d_{\not{D}}} = -\frac{2\pi}{\not{D}!}\int \hat{d}^D q\, \frac{1}{q^2} e^{iq\cdot x} \sum_i J^{(i)}_{d_1\cdots d_{\not{D}}}(q). \tag{4.8}$$

4.1.1 The Hamiltonian for an interacting gas of vortices

Let us assume that the system we are studying consists of a superfluid confined to a volume Ω, with i vortices given by $R^{(i)}(x)$. By substituting the results of Sec. 4.1 into Eq. (4.1), we obtain the total energy in terms of the sources:

$$\begin{aligned}\mathcal{H} &= \frac{1}{2}K\left(\frac{2\pi}{\not{D}!}\right)^2 \epsilon_{d_1\cdots d_D}\,\epsilon_{\bar{d}_1\cdots\bar{d}_{D-1}d_D}\sum_{i,j}\int \hat{d}^D q\, \frac{q_{d_{D-1}} q_{\bar{d}_{D-1}}}{(q\cdot q)^2} \\ &\quad \times e^{-iq\cdot(x-x')} J^{(i)}_{d_1\cdots d_{\not{D}}}(x)\, J^{(j)}_{\bar{d}_1\cdots\bar{d}_{\not{D}}}(x')\end{aligned} \tag{4.9}$$

In Eq. (4.9) we can distinguish between two types of terms: terms for which $i = j$, denoted by $\mathcal{H}_{self}$, which account for the self energy of the vortices – case already treated in Chapter 3; and terms for which $i \neq j$, which give rise to the interaction between pairs of vortices. In evaluating the interaction Hamiltonian, denoted by $\mathcal{H}_{int}$, i.e.,

$$\begin{aligned}\mathcal{H}_{int} &= \frac{1}{2}K\left(\frac{2\pi}{\not{D}!}\right)^2 \epsilon_{d_1\cdots d_D}\,\epsilon_{\bar{d}_1\cdots\bar{d}_{D-1}d_D}\sum_{i,j}\int \hat{d}^D q\, \frac{q_{d_{D-1}} q_{\bar{d}_{D-1}}}{(q\cdot q)^2} \\ &\quad \times e^{-iq\cdot(x-x')} J^{(i)}_{d_1\cdots d_{\not{D}}}(x)\, J^{(j)}_{\bar{d}_1\cdots\bar{d}_{\not{D}}}(x'),\end{aligned} \tag{4.10}$$

we assume that the average distance between two vortices $|x - x'|$, (denoted by X) is much larger than the linear dimension of the vortices themselves. Similar momentum integrals had been computed in Appendix C, thus we obtain:

$$(2\pi)^D \int \hat{d}^D q\, e^{-iq\cdot X}\frac{q_{d_{D-1}} q_{\bar{d}_{D-1}}}{(q\cdot q)^2} = \frac{\mathcal{P}_1}{(X\cdot X)^{\not{D}/2}}\left(\delta_{d_{D-1}\bar{d}_{D-1}} - \not{D}\frac{X_{d_{D-1}} X_{\bar{d}_{D-1}}}{X\cdot X}\right). \tag{4.11}$$

In the process of evaluating the Hamiltonian we have to take into account that we will average over all orientations and positions of the vortices, thus terms containing $X_{d_{D-1}} X_{\bar{d}_{D-1}}$ will average to zero if $d_{D-1} \neq \bar{d}_{D-1}$. The last step in evaluating the interaction Hamiltonian is to contract the skew-symmetric symbols: $\epsilon_{d_1 \cdots d_D} \, \epsilon_{\bar{d}_1 \cdots \bar{d}_{D-1} d_D}$, which yields:

$$\mathcal{H}_{int} = \mathcal{C}_2 \sum_{i \neq j} J^{(i)}_{d_1 \cdots d_{\not{D}}}(x) U(x - x') J^{(j)}_{d_1 \cdots d_{\not{D}}}(x'), \tag{4.12}$$

where $\mathcal{C}_2$ is a numerical constant, given by

$$\mathcal{C}_2 = \frac{1}{2} K \left(\frac{2\pi}{\not{D}!} \right)^2 2\mathcal{P}_1 \frac{1}{(2\pi)^D} \not{D}!, \tag{4.13}$$

and U is the power law interaction between pairs of vortices

$$U(x - x') = \frac{1}{|x - x'|^{\not{D}}}. \tag{4.14}$$

Finally, the total Hamiltonian of the system, which is a sum of of the self energy and interaction terms, is given by

$$\begin{aligned} \mathcal{H}_{eff} &= \mathcal{H}_{self} + \mathcal{H}_{int} \\ &= \sum_i \mathcal{C}_1 K (L^{(i)})^{\not{D}} \ln \frac{L^{(i)}}{\xi} + \sum_{i \neq j} \mathcal{C}_2 K J^{(i)}_{d_1 d_2 \cdots d_{\not{D}}}(x) U(x - x') J^{(j)}_{d_1 d_2 \cdots d_{\not{D}}}(x'). \end{aligned} \tag{4.15}$$

For the self energy terms, the constant $\mathcal{C}_1$ is the volume of the sub-manifold supporting the vortex.

4.2 Scaling equations

4.2.1 Partition function for a gas of vortices

We begin by defining our system to be a collection of vortices confined to the volume Ω. Based on energy considerations, we further assume that the vorticity J of each vortex is ± 1, and that branched vortices are excluded. The partition function this system is given by

$$Z = \sum_{\text{configs}} \mathrm{e}^{-\sum_{i \neq j} \mathcal{H}_{self} + \mathcal{H}_{int}}, \tag{4.16}$$

where $\sum_{\text{configs}}$ is the sum over all possible configurations, i.e.,

$$\sum_{\text{configs}} \cdots = \sum_{N=0}^{\infty} \sum_{J=\pm 1} \frac{1}{N!} \prod_{i}^{N} \int \frac{d^D R}{a^D} \mathcal{A}_{\not{D}-1} \int_a^{\infty} \frac{d\rho}{a} \cdots . \tag{4.17}$$

Here, R and ρ are, respectively, the position of the center, and the radius of the vortex. The quantity a is a length scale of the same order of magnitude as the linear size of the vortices such that it defines a scaling parameter $l = \ln a/\xi$, and consequently we will label the coupling at the scale a via the scaling parameter l.

We further introduce the fugacity y_l, as the measure of the energy cost of a vortex of size (i.e., radius) a using

$$y_l := \mathrm{e}^{\mathcal{C}_1 K (L^a)^{\not{D}} \ln \frac{a}{\xi}}. \tag{4.18}$$

Then, the self-energy terms in the partition function become:

$$\begin{aligned} \mathrm{e}^{-\sum_i \mathcal{C}_1 K (L^{(i)})^{\not{D}} \ln \frac{L^{(i)}}{\xi}} &= \mathrm{e}^{-\sum_i \mathcal{C}_1 K a^{\not{D}} \left(\frac{L^{(i)}}{a}\right)^{\not{D}} \left(\ln \frac{L^{(i)}}{a} + \ln \frac{a}{\xi}\right)} \\ &= y_l^{\sum_i \left(\frac{L^{(i)}}{a}\right)^{\not{D}}} \mathrm{e}^{-\mathcal{C}_1 K a^{\not{D}} \sum_i \left(\frac{L^{(i)}}{a}\right)^{\not{D}} \ln \frac{L^{(i)}}{a}} \end{aligned} \tag{4.19}$$

For convenience we absorb $a^{\not{D}}$ into K, such that at scale a the coupling $K_l \to K_l a^{\not{D}}$. Note that the interaction term in the Hamiltonian remains the same if in addition to absorbing the the $a^{\not{D}}$ term into the coupling K_l we measure the distance between vortices in units of a, i.e.

$$U(x-x') = \left|\frac{x-x'}{a}\right|^{-\not{D}}. \tag{4.20}$$

With these definitions the partition function for a collection of vortices reads:

$$\begin{aligned} Z \;&=\; \sum_{N=0}^{\infty}\sum_{J=\pm 1}\frac{1}{N!}\prod_{i}^{N}\int\frac{d^DR}{a^D}\,\mathcal{A}_{\not{D}-1}\int_a^{\infty}\frac{d\rho}{a}\;y_l^{\sum_i\left(\frac{L^{(i)}}{a}\right)^{\not{D}}}\,\mathrm{e}^{-\mathcal{C}_1K_l\sum_i\left(\frac{L^{(i)}}{a}\right)^{\not{D}}\ln\frac{L^{(i)}}{a}} \\ &\quad\times\mathrm{e}^{-\sum_{i\neq j}\mathcal{C}_2K_l\int d^Dx\int d^Dx'\,J^{(i)}_{d_1d_2\cdots d_{\not{D}}}(x)U(x-x')J^{(j)}_{d_1d_2\cdots d_{\not{D}}}(x')}. \end{aligned} \tag{4.21}$$

Unlike the case of $D = 2$, where the lowest energy excitation is a pair vortex-antivortex, in dimension $D > 2$, all configurations are neutral, therefore we don't have to impose the condition that the sum of all vorticities for a particular configuration is zero.

4.2.2 The renormalization-group procedure and scaling equations

In the previous section we studied the statistical mechanics of fluctuating number of vortices from the perspective of the grand canonical ensemble. As in the two-dimensional case, treated in Chapter 2, we allow the two couplings K_l and yl to depend on the scaling parameter l with the aim of computing scaling equations (i.e., a set of differential equations that relate the change of the couplings with the scaling parameter).

We proceed with an outline of the renormalization-group procedure that we are going to carry in detail in this section. We identify a configuration of vortices at scale a, and we integrate low energy excitations between scales a and $a + da$. Next we change all the explicit scale dependencies from a to $a + da$ in the partition function, keeping only terms linear in the scaling parameter $dl = da/a$. Absorbtion of all changes of order $dl = da/a$ into the renormalized couplings recovers the initial *form* of the partition function with new couplings K_{l+dl} and y_{l+dl}. Assuming that the changes in the couplings are infinitesimal, we derive scaling equations for K_l and y_l.

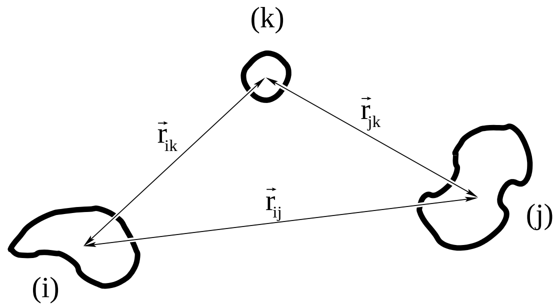

Figure 4.1: Schematic description for the elimination of a low energy excitation. The interaction between the vortices i and j is mediated by vortex k.

The partition function for configurations of vortices at scale a is given by Eq. 4.21, and further denote a vortex with size between a and $a + da$ by k centered at position $R^{(k)}$. To integrate out this excitation, we will single out all the quantities that dependent on vortex k, as follows: in the partition function we separate the integrals involving R_k and ρ_k:

$$\sum_{J=\pm 1} \frac{1}{N!} \prod_{i}^{N} \int \frac{d^D R}{a^D} \mathcal{A}_{\not{D}-1} \int_a^\infty \frac{d\rho}{a} \cdots =$$
$$\sum_{J=\pm 1} \frac{1}{(N-1)!} \left(\prod_{i \neq k}^{N} \int \frac{d^D R}{a^D} \mathcal{A}_{\not{D}-1} \int_a^\infty \frac{d\rho}{a} \right) \int \frac{d^D R_k}{a^D} \mathcal{A}_{\not{D}-1} \int_a^\infty \frac{d\rho_k}{a} \cdots . \quad (4.22)$$

On the other hand, all the interaction and self-energy terms in the Hamiltonian that depend on vortex k are

$$\begin{aligned} Z &= \sum_{config} \mathrm{e}^{-\mathcal{H}(J^{(i\neq k)})} y_l^{\left(\frac{L^{(k)}}{a}\right)^{\not{D}}} \mathrm{e}^{-\mathcal{C}_1 K_l \sum_i \left(\frac{L^{(k)}}{a}\right)^{\not{D}} \ln \frac{L^{(k)}}{a}} \\ &\quad \times \mathrm{e}^{-\mathcal{C}_2 K_l \sum_{i\neq k} J^{(i)}_{d_1 d_2 \cdots d_{\not{D}}}(x) U(x-x_k) J^{(k)}_{d_1 d_2 \cdots d_{\not{D}}}(x_k)}, \end{aligned} \quad (4.23)$$

where $\mathcal{H}(J^{(i\neq k)})$ is the collective hamiltonian of the other $N-1$ vortices, the k^{th}, vortex being excluded.

Using the fact that the radius of vortex k is approximately a, we are essentially left with the computation of the following quantity:

$$\mathcal{J} = \int \frac{d^D R_k}{a^D} y_l \mathcal{A}_{\not{D}-1} \int_a^\infty \frac{d\rho}{a} \, \mathrm{e}^{-\mathcal{C}_2 K_l \sum_{i\neq k} J^{(i)}_{d_1 d_2 \cdots d_{\not{D}}}(x) U(x-x_k) J^{(k)}_{d_1 d_2 \cdots d_{\not{D}}}(x_k)}, \tag{4.24}$$

which we carry in detail in Appendix H. By using integration by parts, with the proper boundary conditions imposed, we obtain:

$$\mathcal{J} \approx \int \frac{d^D R_k}{a^D} \mathcal{A}_{\not{D}-1} y_l dl \prod_{i,j\neq k} \prod_{x,x'} \Bigg(1 + \frac{\not{D}!}{D} \mathcal{C}_2^2 K_l^2 (\mathcal{A}_{D-1})^2 \times J^{(i)}_{d_1 d_2 \cdots d_{\not{D}}}(x) U(R^{(k)} - x)(-\triangle U(R^{(k)} - x')) J^{(j)}_{d_1 d_2 \cdots d_{\not{D}}}(x') \Bigg). \tag{4.25}$$

For future use it will be useful to note that the interaction between two vortices $U(x-x') = \frac{1}{|x-x'|^{\not{D}}}$ is the Green's function of the Laplace operator $-\triangle$ in D dimensions, i.e.,

$$-\triangle U(x-x') = 2\pi \delta^{(D)}(x-x'). \tag{4.26}$$

Thus, by substituting Eq. 4.26 in Eq. 4.25 we obtain for the quantity $\mathcal{J}$ the approximate result:

$$\mathcal{J} \approx \mathcal{A}_{D-1} y_l dl \Bigg(\frac{\Omega}{a^D} + \frac{\not{D}!}{D} \mathcal{C}_2^2 K_l^2 (\mathcal{A}_{D-1})^2 \sum_{i,j} \sum_{x,x'} \times J^{(i)}_{d_1 d_2 \cdots d_{\not{D}}}(x) U(x-x') J^{(j)}_{d_1 d_2 \cdots d_{\not{D}}}(x') \Bigg), \tag{4.27}$$

where $\int d^D R^{(k)} = \Omega$, the total volume occupied by the superfluid.

Initially the system contained N vortices; after the elimination of the k vortex we are left with $N-1$. The quantity we computed, $\mathcal{J}$, scales the $N-1$ term in the partition function, and since N was chosen to be arbitrary, every term in the summation over configurations changes in a similar way. Let us denote the change in the partition function that comes from integrating out vortex k, by $\delta Z^{(k)}$. Because this change is proportional to dl we can approximate Z by

$$Z_{l+dl} = Z_l + \delta Z^{(k)} \approx Z_l(1 + \frac{\delta Z^{(k)}}{Z_l}) \approx Z_l \exp\left[\delta Z^{(k)}/Z_l\right]. \tag{4.28}$$

At this stage, it is straight forward to identify the quantity $\mathcal{J}$ in Eq. (4.24) with $\delta Z^{(k)}/Z_l$.

After eliminating the low energy excitation, in order to put the partition function in the same *form* as the partition function at scale $a + da$, we have to change the explicit scale dependencies in the the sum over configurations

$$\int \frac{d^D R}{a^D} \int_a^\infty \frac{d\rho}{a} \cdots = \int \frac{d^D R}{(a+da)^D} \int_{a+da}^\infty \frac{d\rho}{a+da} (1+dl)^{D+1} \cdots . \tag{4.29}$$

In the definition of coupling K_l at scale a, we absorbed a factor of $a^{\not{D}}$, and consequently K_l is rescaled by a factor of

$$K_l \to K_l(1 - \not{D} dl). \tag{4.30}$$

Adding all the terms from Eqs. (4.27) to (4.30), up to linear terms in dl, we obtain a partition function of the form:

$$\begin{aligned} Z_{l+dl} \approx & \sum_{N=0}^{\infty} \sum_{J=\pm 1} \frac{1}{N!} \prod_i^N \int \frac{d^D R}{(a+da)^D} \int_{a+da}^\infty \frac{d^D \rho}{(a+da)^D} \\ & \times y_l^N (1+dl)^{(D+1)N} \mathrm{e}^{-\mathcal{C}_1 K_l \sum_i \left(\frac{L^{(i)}}{a+da}\right)^{\not{D}} \ln \frac{L^{(i)}}{a+da}} \\ & \times \mathrm{e}^{-\sum_{i \neq j} \mathcal{C}_2 K_l (1 + \not{D} dl - \frac{\not{D}!}{D} (\mathcal{A}_{D-1})^3 \mathcal{C}_2 K_l y_l dl) J^{(i)}_{d_1 d_2 \cdots d_{\not{D}}}(x) U(x-x') J^{(j)}_{d_1 d_2 \cdots d_{\not{D}}}(x')} \\ & \times \mathrm{e}^{\frac{1}{a^D} \mathcal{A}_{D-1} \Omega y_l dl} \mathrm{e}^{\mathcal{O}^2(dl)}. \end{aligned} \tag{4.31}$$

Comparison of Eq. (4.31) with Eq. (4.21) allows us to identify the scaling equations for the couplings. We assume, as expected, that the change of couplings from scale a to $a + da$

is infinitesimal. The scaling equation for K_l can be read directly from Eq. (4.31):

$$dK_l = \left(\not{D} K_l - \frac{\not{D}!}{D} (\mathcal{A}_{D-1})^3 \mathcal{C}_2 K_l^2 y_l \right) dl. \tag{4.32}$$

To compute the scaling equation for the fugacity, first note that the quantities $\sum_i \left(\frac{L^{(i)}}{a+da}\right)^{\not{D}}$ as well as $\sum_i \left(\frac{L^{(i)}}{a+da}\right)^{\not{D}} \ln \frac{L^{(i)}}{a+da}$ are dimensionless and of the same order as the total number of vortices N. In $D = 2$ the equality is exact, because the fugacity has only a logarithmic dependence, but for $D \geq 3$ it is only an approximation. Nevertheless, even within this approximation we show later that we obtain fairly accurate results for critical exponents in $D = 3$, and similarly, with the values predicted in $D = 4$. With all the corresponding summations of order N, we collect all the terms dependent on the fugacity from Eq. (4.31)

$$(y_l + dy_l)^N = y_l^N (1 + dl)^{(D+1)N} \, \mathrm{e}^{-\not{D} \mathcal{C}_1 K_l N dl}, \tag{4.33}$$

which yields the following scaling equation for y_l:

$$dy_l \quad = \quad \left(D + 1 - \not{D} \mathcal{C}_1 K_l\right) y_l \, dl \tag{4.34}$$

Lastly, the term dependent of the total volume Ω, scales the free energy density, defined as $f = \ln Z/\Omega$, yielding the following relation

$$df_l = \frac{1}{a^D} \mathcal{A}_{D-1} y_l dl. \tag{4.35}$$

4.3 Linearization of the flow equations

4.3.1 Fixed points of the flow equations

In this section we investigate the existence of fixed points –points in the coupling spaces at which the derivative of the couplings with respect to the scaling parameter vanish. Written in differential form, the flow equations, read as follows:

$$\frac{dK_l}{dl} = \not{D} K_l - AK_l^2 y_l, \tag{4.36}$$

$$\frac{dy_l}{dl} = \left(D + 1 - -\not{D}\mathcal{C}_1 K_l\right) y_l, \tag{4.37}$$

$$\frac{df_l}{dl} = a^{-D}\mathcal{A}_{D-1} y_l, \tag{4.38}$$

where A and B are dimension-dependent coefficients having the values

$$A = \frac{\not{D}!}{D2^D}(\mathcal{A}_{D-1})^3\mathcal{C}_2, \qquad B = \mathcal{C}_1, \tag{4.39}$$

where $\mathcal{C}_2$ and $\mathcal{C}_1$ are given in Eqs. (4.13, 4.15). These values depend on non-universal quantities (such as the volumes of the sub-manifolds supporting the vortex), however, we shall see later in this chapter, that the universal characteristics of the transition do not depend on A and B, instead depending only on the dimension of the embedding space D.

The fixed points for the flow equations, i.e., Eqs. (4.36, 4.37), are points in the parameter space for which the derivatives with respect to the scaling parameter l vanish. The system of equations thus obtained admits trivial solutions – with at least one of the parameters being equal to zero. The first trivial solution is $(K_l^{high}, y_l^{high}) = (0, y_0 e^{(D+1)l})$, where y_0 is the fugacity of the scale a_0, the initial point for the iteration of the flow equations. Recalling the definition of K, i.e., Eq. (4.2), we denote this trivial fixed point as the high temperature fixed point. The second trivial solution is $(K_l^{low}, y_l^{low}) = (K_0 e^{\not{D}l}, 0)$, and we denote it accordingly, as the low temperature fixed point. Both trivial fixed points are stable, thus, assuming that the topology of the renormalization-group flow is one dimensional (i.e., the flow towards different regimes is dependent on the temperature), there must be another, unstable, non-trivial fixed point at non-zero temperature. Indeed, the non-trivial solution is given by the following values for the couplings

$$\begin{aligned} K_l^* &= \frac{D+1}{(D-2)BL_l}, \\ y_l^* &= \frac{(D-2)^2 B}{(D+1)A}. \end{aligned} \tag{4.40}$$

4.4 Critical exponents

We begin this section by defining [3, 10] a few critical exponents which we compute for a superfluid in arbitrary dimension. The heat capacity exponent α characterizes the behavior of the specific heat near the critical point

$$C \sim |t|^{-\alpha}, \tag{4.41}$$

where t is the reduced temperature $t = (T - T_c)/T_c$. Similarly, the spatial correlation exponent ν describes the behavior of spatial correlations ζ near the critical point

$$\zeta \sim |t|^{-\nu}. \tag{4.42}$$

Having established the existence of fixed points for the flow equations, in the next section we analyze the flow in vicinity of the non-trivial fixed point. To do so, we exchange the parameters K_l and y_l for the parameters $\tilde{K}$ and $\tilde{y}$, which are defined in terms of the fixed-point values via

$$K_l = K_l^*(1 + \tilde{K}_l) \qquad y_l = y_l^*(1 + \tilde{y}_l). \tag{4.43}$$

Thus, $\tilde{K}$ and $\tilde{y}$ nearly describe small departures the fixed point values (K_l^*, y_l^*). Substituting these definitions in Eqs. (4.36, 4.37), we obtain a new set of flow equations

$$\frac{d}{dl}\begin{pmatrix} \tilde{y}_l \\ \tilde{K}_l \end{pmatrix} = \begin{pmatrix} 0 & -(D+1) \\ -\not{D} & -\not{D} \end{pmatrix} \begin{pmatrix} \tilde{y}_l \\ \tilde{K}_l \end{pmatrix}. \tag{4.44}$$

We denote the eigenvalues of the matrix in Eq. (4.44), by λ_+ and λ_-; regardless of the dimensionality of the embedding space D, these eigenvalues have opposite signs, hence our choice of indexing them accordingly. The corresponding eigenvectors, u_+ and u_-, define the relevant/irrelevant axes in the (K_l, y_l) plane that become increasingly important or unimportant as we iterate the flow equations. Of the two eigenvectors only u_+, is relevant and, following from the definition of K, in Eq. (4.2), a temperature field being proportional to the reduced temperature $u_+ \sim (T - T_c)/T_c$.

We initiate the RG flow at a point where the values for the couplings are K_0 and y_0. Given the flow equations derived in the previous section, Eqs. (4.36, 4.37), we can write the partition function as

$$Z(K_0, y_0) = e^{-(f_l - f_0)} Z(K_l, y_l) = e^{-(f_l - f_0)} Z(u_+ e^{\lambda_+ l}, u_- e^{\lambda_- l}). \tag{4.45}$$

We stop iterating the flow equations at the critical point, where the scale parameter is of order $l = \ln \zeta/\xi$. The behavior of the partition function near the critical point is obtained by taking the limit $l^* = \lim_{T \to T_c} l$ to infinity, in which case the partition function is well behaved only if $\nu = 1/\lambda_+$ For arbitrary dimension D it is given by

$$\nu = \frac{2}{2 - D + \sqrt{(D-2)(5D+2)}}. \tag{4.46}$$

In Table 4.1 we present, for different values of D, numerical values for the correlation exponent ν. In $D = 3$ we obtain reasonable agreement with the accepted value, but in $D = 4$ we obtain a rather smaller value that it is known to hold.

ν	$D = 3$	$D = 4$
loop renormalization	0.64	0.43
expected values	0.66	0.50

Table 4.1: A comparison for numerical values of the critical exponent ν, for different values of D.

it gives very good agreement with the expected values, while in $D \geq 4$ we obtain a slightly smaller value. In our computation of the flow equation for the fugacity y_l we used a series of approximations that may affect the eigenvalues of matrix in Eq. (4.44). We have also assumed that the vortices are smooth manifolds, even near the critical point which, as a matter of fact, does not hold true. Our result for the critical exponent ν, should thus be interpreted as a *lower bound*. The specific heat exponent α, can be obtained from ν using the Josephson scaling relation $\alpha = 2 - D\nu$. A feature that the present approach seems to capture is the vanishing of the superfluid density as transition temperature is approached from the low temperature side. Recalling that the superfluid density is related to the coupling K via $\rho_l = e^{\not{D} l} K_l$ we can obtain the flow equation for the superfluid density:

$$\frac{d\rho_l}{dl} = A e^l y_l \rho_l^2. \tag{4.47}$$

This can be integrated from $T = 0$ to $T = T_c$ and using the fact [29] that at $T = 0$ the superfluid density is of order unity, obtaining

$$\frac{1}{\rho_c} - 1 = \int_{\ln a_0/\xi}^{\infty} A e^l y_l \tag{4.48}$$

which diverges as $l \to \infty$. Thus we see that the superfluid density vanishes at the critical point.

Chapter 5

Concluding remarks

We have explored the extension, to arbitrary spatial dimension, of Onsager-Feynman quantized vortices in the flow of a superfluid, focusing on the structure and energetics of the associated superflow and the corresponding dynamics of the vortices. To do this, we have analyzed the superflow that surrounds the vortices at equilibrium, by invoking an extension to arbitrary dimensions of the three-dimensional analogy between: (i) the magnetic field of Ampère-Maxwell magnetostatics in the presence of specified, conserved electric current concentrated on infinitesimally thin loops; and (ii) the velocity field of equilibrium superfluidity of helium-four in the presence of specified, conserved vorticity concentrated on infinitesimally thin vortices.

By constructing the appropriate conditions that the velocity field must obey at equilibrium, if the flow is to have a given vortical content, we have determined the corresponding flow in terms of this vortical content, via the introduction of a suitable, skew-symmetric gauge potential, the rank of which is two fewer than the spatial dimension. The structure of the flows associated with generically shaped vortices has enabled us to develop results for the dynamics that vortices inherit from these flows in all dimensions, and for the way in which this dynamics reflects both the intrinsic and extrinsic geometry of the vortices when the vortices are large and smooth, relative to the vortex core size. This structure has also allowed us to ascertain the velocities with which the higher-dimensional extensions of circular vortex rings (i.e., vortex spheres and hyper-spheres) propagate, as well as to analyze the higher-dimensional extension (to planes and hyper-planes) of the Kelvin problem of the vibrations of a line vortex.

For all dimensions, we have expressed the equilibrium kinetic energy of superflow in the presence of a generic vortex in terms of the shape of the vortex, and have identified the leading contribution to this energy—for vortices that are smooth and large, relative to the vortex core size—as being proportional to the volume of the sub-manifold supporting the vortex. We have also computed the kinetic energy for the higher-dimensional extensions of vortex rings, and used these energies to determine the momenta, and energy-momentum relations, of these maximally symmetrical vortices. In addition, we have shown how the structure of our results, if not their details, can be obtained via elementary arguments, including scaling analyses.

The geometry of the vortices—both the intrinsic geometry of their shapes and the extrinsic geometry of the manner in which the vortices are embedded in the higher-dimension ambient space—plays a central role in our developments. The natural language for such developments is that of exterior calculus and differential forms, but we have not made this language essential, using instead the equivalent, but more widely known, language of skew-symmetric tensor fields.

We have employed the same type of formalism to construct the Hamiltonian for a collection of vortices. Besides the self energies of the vortices, this Hamiltonian has interactions that, in the limit of large separations between the vortices (when compared to their linear dimensions), are a simple power-law form. We have employed renormalization-group techniques, to integrate out low-energy excitations, with the goal of computing critical exponents for the normal-to-superfluid phase transition. The results we thus obtain for the correlation length exponent ν of 0.64 is in reasonable agreement with the expected value of 0.66 for $D = 3$. For $D = 4$ and higher, our result for ν should be interpreted as a lower bound being smaller than the known value of 0.5. Finally our approach indicates that the superfluid density vanishes near the critical temperature for any dimension $D \geq 3$, in agreement with experiments in three dimensional superfluid helium. For the normal-to-superfluid transition the inclusion of a magnetic field term may be of interest; in this way we can include a magnetic scaling field whose eigenvalue will determine the other critical coefficients.

Further issues that one might wish to consider within the present framework include the dynamics of lattices of higher-dimensional extensions of vortices associated with equilibrium states of rotating superfluids, along with the collective vibrations of such lattices (i.e., Tkachenko modes), as well as Magnus-type forces on vortices provided by background flows. One might also consider the energetics and motion of knotted vortices in three dimensions (cf. Kelvin's "vortex atoms" [38]) and—where mathematically available—topologically more exotic, higher-dimensional vortices, including higher-genus surfaces. Still more challenging would be the generalization of the circle of ideas discussed in this paper to the non-linear setting provided by non-Abelian gauge fields.

Appendix A

Source terms and quantized circulation

In this appendix, we discuss two properties of the source field J. First, we show that the circulation, Eq. (3.2), of the superflow around the vortex sub-manifold associated with the source term given in Eq. (3.11) is quantized to unit value. We then show that the source field is divergenceless.

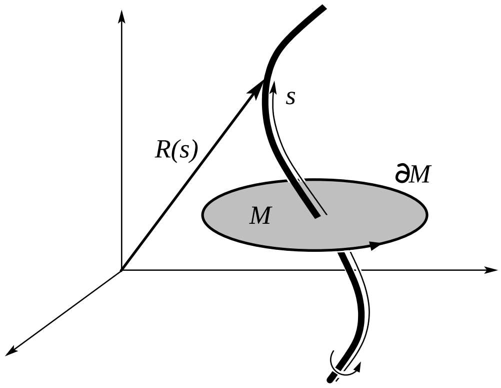

Figure A.1: A closed path $\partial\mathcal{M}$ around a segment of a vortex loop specified by $R(s)$, with arc-length parameter s. $\partial\mathcal{M}$ is the boundary of the two-dimensional region $\mathcal{M}$, which is punctured by the $\not{D}$-dimensional vortex.

We begin by considering a closed, one-dimensional path $\partial\mathcal{M}$, centered on the vortex sub-manifold and encircling it, along with the associated two-dimensional region $\mathcal{M}$ bounded by $\partial\mathcal{M}$ and is punctured by the vortex sub-manifold, as shown in Fig. A.1. In the language of differential forms, Stokes' theorem, applied to Eq. (3.9), determines that the circulation κ

around $\partial\mathcal{M}$ is given by

$$\begin{aligned} \kappa &= \oint_{\partial\mathcal{M}} V_d(x)\, dx_d = \oint_{\partial\mathcal{M}} V && \text{(A.1)} \\ &= \int_{\mathcal{M}} dV = 2\pi \int_{\mathcal{M}} \star J = \frac{2\pi}{2!} \int_{\mathcal{M}} (\star J)_{d_1 d_2}(x)\, dx_{d_1} \wedge dx_{d_2}\,. && \text{(A.2)} \end{aligned}$$

We take $\partial\mathcal{M}$ to be circular, with the points on it being given by

$$x = R(s_0) + a_1 N^1 + a_2 N^2, \tag{A.3}$$

where a_1 and a_2 vary subject to the constraint $a_1^2 + a_2^2 = \text{const.} > 0$, and N^1 and N^2 are any pair of mutually perpendicular unit vectors that are perpendicular to the vortex sub-manifold at the point s_0. An elementary computation then shows that $dx_{d_1} \wedge dx_{d_2} = da_1 \wedge da_2\, (N^1 \wedge N^2)_{d_1 d_2}$, so that, from Eq. (A.2), the circulation becomes

$$\kappa = \frac{2\pi}{2!} \int_{\mathcal{M}} (\star J)_{d_1 d_2}(x)\, (N^1 \wedge N^2)_{d_1 d_2}\, da_1 \wedge da_2. \tag{A.4}$$

Next, we insert the explicit components $\star J_{d_1 d_2}$, Eq. (3.13), for the specific form of $\mathcal{M}$ defined by Eq. (A.3), to arrive at

$$\begin{aligned} \kappa = \frac{2\pi}{2!} \int_{\mathcal{M}} \int d^{\not{D}} s\, \epsilon_{d_1 \cdots d_{\not{D}} d_{D-1} d_D} (\partial_{a_1} R_{d_1}) \cdots (\partial_{a_{\not{D}}} R_{d_{\not{D}}})\, (N^1 \wedge N^2)_{d_{D-1} d_D} \\ \times \delta^{(D)} \big(R(s_0) + a_1 N^1 + a_2 N^2 - R(s) \big)\; da_1 \wedge da_2. \qquad \text{(A.5)} \end{aligned}$$

Next, we Taylor-expand $R(s)$ in the argument of the delta function about the point $s = s_0$ and use the orthogonality of N^1 and N^2 with the tangent vectors $\{\partial R / \partial s_a\}_{a=1}^{\not{D}}$ at s_0 to obtain

$$
\begin{aligned}
\kappa \;\; &= \;\; \frac{2\pi}{2!}\epsilon_{d_1\cdots d_{\not{D}}d_{D-1}d_D}(\partial_{a_1}R_{d_1})\cdots(\partial_{a_{\not{D}}}R_{d_{\not{D}}})\,(N^1\wedge N^2)_{d_{D-1}d_D}\Big|_{s=s_0} \\
&\qquad\qquad \times\int d^{\not{D}}s\,\delta^{(D-2)}\Big((s-s_0)_a\,\partial_a R(s)|_{s=s_0}\Big) \\
&\qquad\qquad\qquad\qquad \times\int\delta^{(1)}(a_1)\,da_1\int\delta^{(1)}(a_2)\,da_2\,.
\end{aligned}
\tag{A.6}
$$

The last two integrals each give a factor of unity. The remaining one, over s, is readily seen to give a factor of $1/|\det(\partial_a R)|$ provided we invoke the fact that the term $\partial_a R$ is a square $\not{D}\times\not{D}$ matrix in the subspace of vectors tangent to the vortex sub-manifold. As we discuss in Section 3.2, the factor $\epsilon_{d_1\cdots d_{\not{D}}d_{D-1}d_D}(\partial_{a_1}R_{d_1})\cdots(\partial_{a_{\not{D}}}R_{d_{\not{D}}})$ is given by $(N^1\wedge N^2)_{d_{D-1}d_D}\,\det(\partial_a R)$. Hence, in the formula for the circulation the determinants cancel, and the remaining two factors of the form $N^1\wedge N^2$ contract with one another to give a factor of 2, so that we arrive at the result

$$
\kappa = \pm 1, \tag{A.7}
$$

the sign depending on the sense in which the encircling path $\partial\mathcal{M}$ is followed. Thus, we see that the circulation of the superflow around the vortex sub-manifold associated with the source term given in Eq. (3.11) is indeed quantized to unit value.

In order to discuss the divergencelessness property of J, we should distinguish between two cases. In the case of $D=3$ the source field is a vector, which consequently has no skew-symmetric properties. In arbitrary dimension D greater than or equal to 4, the source field J has a tensorial structure that we exploit in our analysis. Let us consider first the case case of $D=3$. The source term is then given by

$$
J_{d_1}(x) = \int ds\frac{dR_{d_1}(s)}{ds}\int\frac{d^3q}{(2\pi)^3}e^{-iq\cdot(x-R(s))}, \tag{A.8}
$$

and its divergence may be manipulated as follows:

$$\begin{aligned}
\nabla_{d_1} J_{d_1}(x) &= \int ds \frac{dR_{d_1}(s)}{ds} \int \frac{d^3q}{(2\pi)^3}(-iq_{d_1})e^{-iq\cdot(x-R(s))} \\
&= \int ds \int \frac{d^3q}{(2\pi)^3}(-iq_{d_1})\frac{dR_{d_1}(s)}{ds}e^{-iq\cdot(x-R(s))} \\
&= \int \frac{d^3q}{(2\pi)^3} \int ds \, \frac{d}{ds}\left(e^{-iq\cdot(x-R(s))}\right). \qquad \text{(A.9)}
\end{aligned}$$

The integrand in the last line of Eq. (A.9) is a total derivative of a periodic function of s, and therefore the integral vanishes.

For $D \geq 4$ the source term has the structure

$$J_{d_1\cdots d_{\not{D}}}(x) = \int d^{\not{D}} s \, \varepsilon_{a_1\cdots a_{\not{D}}}(\partial_{a_1}R_{d_1})\cdots(\partial_{a_{\not{D}}}R_{d_{\not{D}}}) \int \frac{d^Dq}{(2\pi)^D}\, e^{-iq\cdot(x-R(s))}, \qquad \text{(A.10)}$$

and its divergence may be manipulated as follows:

$$\begin{aligned}
\nabla_{d_1} J_{d_1\cdots d_{\not{D}}}(x) = &\int d^{\not{D}} s \, \varepsilon_{a_1\cdots a_{\not{D}}}(\partial_{a_1}R_{d_1})\cdots(\partial_{a_{\not{D}}}R_{d_{\not{D}}}) \\
&\times \int \frac{d^Dq}{(2\pi)^D}(-iq_{d_1})e^{-iq\cdot(x-R(s))} \\
= &\int d^{\not{D}} s \int \frac{d^Dq}{(2\pi)^D}\, \varepsilon_{a_1\cdots a_{\not{D}}}\left[\partial_{a_1}\left(e^{-iq\cdot(x-R(s))}\right)\right](\partial_{a_2}R_{d_2})\cdots(\partial_{a_{\not{D}}}R_{d_{\not{D}}}) \\
= &\int \frac{d^Dq}{(2\pi)^D} \int d^{\not{D}} s \Big\{\varepsilon_{a_1\cdots a_{\not{D}}}\partial_{a_1}\left[e^{-iq\cdot(x-R(s))}(\partial_{a_2}R_{d_2})\cdots(\partial_{a_{\not{D}}}R_{d_{\not{D}}})\right] \\
&-e^{-iq\cdot(x-R(s))}\varepsilon_{a_1\cdots a_{\not{D}}}\partial_{a_1}\left[(\partial_{a_2}R_{d_2})\cdots(\partial_{a_{\not{D}}}R_{d_{\not{D}}})\right]\Big\}. \qquad \text{(A.11)}
\end{aligned}$$

In the last line of Eq. (A.11) the first term vanishes because the integrand is a total derivative of a periodic function, whilst the second term vanishes via the skew-symmetric properties of Levi-Civita symbol. Hence, we have the divergencelessness condition $\nabla_{d_1} J_{d_1\cdots d_{\not{D}}} = 0$.

Appendix B

Computing the velocity field

As shown in Section 3.3.2, the velocity field due to a single, $\not{D}$-dimensional vortex is given by

$$
\begin{aligned}
V_{d_D}(x) \;=\; & -\frac{2\pi}{\not{D}\,!}\,\epsilon_{d_1\cdots d_D}\int \hat{d}^D q\,\frac{iq_{d_{D-1}}}{q\cdot q}\mathrm{e}^{-iq\cdot x} \\
& \times\int d^{\not{D}} s\,\varepsilon_{a_1\cdots a_{D-2}}(\partial_{a_1}R_{d_1})\cdots(\partial_{a_{\not{D}}}R_{d_{\not{D}}})\,\mathrm{e}^{iq\cdot R(s)}. \qquad \text{(B.1)}
\end{aligned}
$$

Performing the q integration using Eq. (C.2) of C, and noting the formula for the surface area of a D-dimensional sphere of unit radius [1], the velocity field becomes

$$
\begin{aligned}
V_{d_D}(x) \;=\; & -\frac{2\pi}{\not{D}\,!}\,\epsilon_{d_1\cdots d_D}\int d^{\not{D}} s\,\varepsilon_{a_1\cdots a_{\not{D}}}(\partial_{a_1}R_{d_1})\cdots(\partial_{a_{\not{D}}}R_{d_{\not{D}}}) \\
& \times\int \hat{d}^D q\,\frac{iq_{d_{D-1}}}{q\cdot q}\mathrm{e}^{iq\cdot(R(s)-x)} \\
=\; & \frac{2\pi}{\not{D}\,!}\,\epsilon_{d_1\cdots d_D}\int d^{\not{D}} s\,\varepsilon_{a_1\cdots a_{\not{D}}}(\partial_{a_1}R_{d_1})\cdots(\partial_{a_{\not{D}}}R_{d_{\not{D}}}) \\
& \times\partial_{d_{D-1}}\int \hat{d}^D q\,\frac{\mathrm{e}^{iq\cdot(R(s)-x)}}{q\cdot q} \\
=\; & \frac{\pi^{\frac{3}{2}-D}}{\not{D}\,!\,4}\mathcal{A}_{D-1}\,\Gamma\!\left(\tfrac{D}{2}-\tfrac{1}{2}\right)\Gamma\!\left(\tfrac{\not{D}}{2}\right)\epsilon_{d_1\cdots d_D} \\
& \times\int d^{\not{D}} s\,\varepsilon_{a_1\cdots a_{\not{D}}}(\partial_{a_1}R_{d_1})\cdots(\partial_{a_{\not{D}}}R_{d_{\not{D}}})\,\left(\partial_{d_{D-1}}\big|x-R(s)\big|^{-\not{D}}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\pi^{\frac{3}{2}-D}}{(\not{D}-1)!\,4}\mathcal{A}_{D-1}\,\Gamma\big(\tfrac{D}{2}-\tfrac{1}{2}\big)\,\Gamma\big(\tfrac{\not{D}}{2}\big)\,\epsilon_{d_1\cdots d_D} \\
&\qquad \times \int d^{\not{D}}s\,\varepsilon_{a_1\cdots a_{\not{D}}}(\partial_{a_1}R_{d_1})\cdots(\partial_{a_{\not{D}}}R_{d_{\not{D}}})\,\frac{\big(x-R(s)\big)_{d_{D-1}}}{\big|x-R(s)\big|^D} \\
&= \frac{1}{2\,\pi^{\not{D}/2}}\,\frac{\Gamma(\not{D}/2)}{(\not{D}-1)!}\,\epsilon_{d_1\cdots d_D} \\
&\qquad \times \int d^{\not{D}}s\,\varepsilon_{a_1\cdots a_{\not{D}}}(\partial_{a_1}R_{d_1})\cdots(\partial_{a_{\not{D}}}R_{d_{\not{D}}})\,\frac{\big(x-R(s)\big)_{d_{D-1}}}{\big|x-R(s)\big|^D}.
\end{aligned}
\tag{B.2}
$$

Appendix C

Invariant integrals

Consider the q integral in Eq. (3.64). Writing X for $(R(s') - R(s))$, and invoking rotational symmetry and dimensional analysis, we have

$$(2\pi)^D \int \hat{d}^D q \, \mathrm{e}^{iq\cdot X} \frac{q_{d'}\, q_d}{(q\cdot q)^2} = \frac{1}{(X\cdot X)^{\not{D}/2}} \left(\delta_{d'd}\, \mathcal{P}_1 + \frac{X_{d'}\, X_d}{X\cdot X}\, \mathcal{P}_2 \right). \tag{C.1}$$

Contracting first with δ and then $X_{d'}\, X_d$, we arrive at the invariant integrals

$$\int d^D q \, \mathrm{e}^{iq\cdot X} \frac{1}{q\cdot q} = \frac{D\mathcal{P}_1 + \mathcal{P}_2}{(X\cdot X)^{\not{D}/2}}, \tag{C.2}$$

$$\frac{1}{(X\cdot X)} \int d^D q \, \mathrm{e}^{iq\cdot X} \frac{(q\cdot X)^2}{(q\cdot q)^2} = \frac{\mathcal{P}_1 + \mathcal{P}_2}{(X\cdot X)^{\not{D}/2}}. \tag{C.3}$$

The integrals (C.2) and (C.2) can be computed in hyper-spherical coordinates, and we thus obtain for $\mathcal{P}_1$ and $\mathcal{P}_2$:

$$\mathcal{P}_1 = \mathcal{A}_{D-1}\, \sqrt{\pi}\, \Gamma\left(\tfrac{D}{2} - \tfrac{1}{2}\right) 2^{D-4}\, \Gamma\left(\tfrac{D}{2} - 1\right), \tag{C.4}$$

$$\mathcal{P}_2 = (2-D)\, \mathcal{A}_{D-1}\, \sqrt{\pi}\, \Gamma\left(\tfrac{D}{2} - \tfrac{1}{2}\right) 2^{D-4}\, \Gamma\left(\tfrac{D}{2} - 1\right), \tag{C.5}$$

where $\mathcal{A}_D$ is the surface area of a D-dimensional sphere of unit radius [1], and $\Gamma(z)$ is the standard Gamma function [25].

Appendix D

Asymptotic analysis for the velocity of a vortex

To determine the asymptotic behaviour of the velocity, we start with Eq. (3.40), which contains the factor

$$\int d^{\not{D}} s \, \varepsilon_{a_1 \cdots a_{\not{D}}} \partial_{a_1} R_{d_1}(s) \cdots \partial_{a_{\not{D}}} R_{d_{\not{D}}}(s) \, \frac{\big(R(\Sigma) - R(s)\big)_{d_{D-1}}}{\big|R(\Sigma) - R(s)\big|^{D}}, \tag{D.1}$$

and Taylor-expand the integrand about the point Σ to obtain

$$\begin{aligned}
\int d^{\not{D}} s \Bigg\{ & \Big[(\partial_{a_1} R_{d_1}) \cdots (\partial_{a_{\not{D}}} R_{d_{\not{D}}}) \big|_{s=\Sigma} \\
& \quad + (\partial_{a_1} R_{d_1}) \cdots (\partial_b \partial_{a_m} R_{d_m}) \cdots (\partial_{a_{\not{D}}} R_{d_{\not{D}}}) \big|_{s=\Sigma} (s_b - \Sigma_b) + \cdots \Big] \\
& \times \Big[- (\partial_b R_{d_{D-1}}) \big|_{s=\Sigma} (s_b - \Sigma_b) \\
& \quad - \frac{1}{2} (\partial_b \partial_{b'} R_{d_{D-1}}) \big|_{s=\Sigma} (s_b - \Sigma_b)(s_{b'} - \Sigma_b) + \cdots \Big] \\
& \times \Bigg(\Big[- (\partial_b R_{\bar{d}}) \big|_{s=\Sigma} (s_b - \Sigma_b) \\
& \quad - \frac{1}{2} (\partial_b \partial_{b'} R_{\bar{d}}) \big|_{s=\Sigma} (s_b - \Sigma_b)(s_{b'} - \Sigma_{b'}) + \cdots \Big] \\
& \times \Big[- (\partial_b R_{\bar{d}}) \big|_{s=\Sigma} (s_b - \Sigma_b) \\
& \quad - \frac{1}{2} (\partial_b \partial_{b'} R_{\bar{d}}) \big|_{s=\Sigma} (s_b - \Sigma_b)(s_{b'} - \Sigma_{b'}) + \cdots \Big] \Bigg)^{-\frac{D}{2}}
\end{aligned} \tag{D.2}$$

Next, we expand the terms in square brackets, retaining only terms to leading order, and collect terms of similar structure in the integration variables s, observing that two such structures arise:

$$\mathcal{I}^{(2)}_{a_1a_2}(\Sigma) \;:=\; \int d^{\not D}s\,\frac{(s_{a_1}-\Sigma_{a_1})(s_{a_2}-\Sigma_{a_2})}{\left[(s_a-\Sigma_a)\,g_{aa'}(\Sigma)\,(s_{a'}-\Sigma_{a'})\right]^{D/2}}, \tag{D.3}$$

$$\mathcal{I}^{(4)}_{a_1a_2a_3a_4} \;:=\; \int d^{\not D}s\,\frac{(s_{a_1}-\Sigma_{a_1})(s_{a_2}-\Sigma_{a_2})(s_{a_3}-\Sigma_{a_3})(s_{a_4}-\Sigma_{a_4})}{\left[(s_a-\Sigma_a)\,g_{aa'}(\Sigma)\,(s_{a'}-\Sigma_{a'})\right]^{(D/2)+1}}. \tag{D.4}$$

The asymptotic behaviours of these structures are given by

$$\mathcal{I}^{(2)}_{a_1a_2}(\Sigma) \;\approx\; \frac{\mathcal{A}_{\not D}}{\not D}\,\frac{\bar{g}_{a_1a_2}}{\sqrt{\det g}}\;\ln(L/\xi), \tag{D.5}$$

$$\mathcal{I}^{(4)}_{a_1a_2a_3a_4}(\Sigma) \;\approx\; \frac{\mathcal{A}_{\not D}}{D\not D}\,\frac{\bar{g}_{a_1a_2}\,\bar{g}_{a_3a_4}+\bar{g}_{a_1a_3}\,\bar{g}_{a_2a_4}+\bar{g}_{a_1a_4}\,\bar{g}_{a_2a_3}}{\sqrt{\det g}}\;\ln(L/\xi), \tag{D.6}$$

where $\bar{g} := g^{-1}$, and g and ḡ are evaluated at Σ.

Appendix E

Geometric content of the vortex velocity

Equation (3.41) for the asymptotic approximation to the vortex velocity $U_{d_D}(\Sigma)$ contains three terms,

$$
\begin{aligned}
\mathcal{J}^1_{d_D} &:= \epsilon_{d_1\cdots d_D}\epsilon_{a_1\cdots a_{\not{D}}}(\partial_{a_1}R_{d_1})\cdots(\partial_{a_{\not{D}}}R_{d_{\not{D}}})(\partial_b\partial_{b'}R_{d_{D-1}})\bar{g}_{bb'}\,, && \text{(E.1)}\\
\mathcal{J}^2_{d_D} &:= \epsilon_{d_1\cdots d_D}\epsilon_{a_1\cdots a_{\not{D}}}(\partial_{b'}R_{d_{D-1}})(\partial_{a_1}R_{d_1})\cdots \\
&\qquad\qquad \times(\partial_b\partial_{a_m}R_{d_m})\cdots(\partial_{a_{\not{D}}}R_{d_{\not{D}}})\bar{g}_{bb'}\,, && \text{(E.2)}\\
\mathcal{J}^3_{d_D} &:= \epsilon_{d_1\cdots d_D}\epsilon_{a_1\cdots a_{\not{D}}}(\partial_b R_{\bar{d}})(\partial_{\bar{b}}\partial_{\bar{b}'}R_{\bar{d}})(\partial_{a_1}R_{d_1})\cdots(\partial_{a_{\not{D}}}R_{d_{\not{D}}})(\partial_{b'}R_{d_{D-1}}) \\
&\qquad\qquad \times\big(\bar{g}_{b\bar{b}}\bar{g}_{b'\bar{b}'}+\bar{g}_{b\bar{b}'}\bar{g}_{\bar{b}\bar{b}'}+\bar{g}_{bb'}\bar{g}_{\bar{b}\bar{b}'}\big), && \text{(E.3)}
\end{aligned}
$$

which we consider in turn.

The first term contains the ingredient

$$
\epsilon_{d_1\cdots d_D}\,\varepsilon_{a_1\cdots a_{\not{D}}}\,(\partial_{a_1}R_{d_1})\cdots(\partial_{a_{\not{D}}}R_{d_{\not{D}}}). \tag{E.4}
$$

which is identical to the term in Eq. (3.14), computed in Section 3.2. By substituting its value into Eq. (E.1), we obtain

$$
\mathcal{J}^1_{d_D} = \not{D}!\,\sqrt{\mathrm{det}g}\;N_{d_{D-1}d_D}\,(\partial_b\partial_{b'}R_{d_{D-1}})\,\bar{g}_{bb'}\,. \tag{E.5}
$$

It is straightforward to observe (via partial integration) that $\mathcal{J}^2$ is identical to $\mathcal{J}^1$, up to a sign. As for $\mathcal{J}^3_{d_D}$, it is proportional to

$$\epsilon_{d_1\cdots d_D}\,\varepsilon_{a_1\cdots a_{\not{D}}}(\partial_b R_{\bar{d}})(\partial_{a_1} R_{d_1})\cdots(\partial_{a_{\not{D}}} R_{d_{\not{D}}})(\partial_{b'} R_{d_{D-1}}), \tag{E.6}$$

a term that is skew-symmetric in the indices $(d_1, \ldots, d_{\not{D}})$ and symmetric under the interchange of $\bar{d}$ and d_i. With the only possible values for both $\bar{d}$ and d_i being in the set $\{1, \ldots, \not{D}\}$, $\mathcal{J}^3$ vanishes.

Appendix F

Velocity for a weak distortion of a hyper-planar vortex

In this appendix, we consider weak distortions of a $D\!\!\!/$-dimensional hyper-planar vortex, and determine the velocity that the corresponding superflow confers on such distortions. We begin with Eq. (3.52), which is Eq. (3.40) but with the replacement Eq. (3.51), which yields the velocity $U(\sigma)$ of points σ on the vortex in terms of H. We then expand U in powers of H, which we take to be small, and, for the sake of compactness, we temporarily omit an overall factor of $2\pi D\!\!\!/\Gamma\left(D\!\!\!//2\right)\big/4\pi^{D/2}$ which multiplies the velocity. Hence, we arrive at the formula

$$\begin{aligned} U_{d_D}(\sigma) &= \epsilon_{d_1\cdots d_D} \int d^{\not D} s\, \partial_{a_1}(F_{d_1}(s)+H_{d_1}(s)) \cdots \partial_{a_{\not D}}(F_{d_{\not D}}(s)+H_{d_{\not D}}(s)) \\ &\qquad \times \frac{\big(F(\sigma)-F(s)+H(\sigma)-H(s)\big)_{d_{D-1}}}{\big|F(\sigma)-F(s)+H(\sigma)-H(s)\big|^D} \qquad \text{(F.1)} \\ &= \epsilon_{d_1\cdots d_D} \int d^{\not D} s\, \partial_{a_1}F_{d_1}(s)\cdots \partial_{a_{\not D}}F_{d_{\not D}}(s) \frac{\big(F(\sigma)-F(s)\big)_{d_{D-1}}}{\big|F(\sigma)-F(s)\big|^D} \\ &\quad +\epsilon_{d_1\cdots d_D} \sum_{\nu=1}^{\not D} \int d^{\not D} s\, \partial_{a_1}F_{d_1}(s)\cdots \partial_{a_\nu}H_{d_\nu}(s)\cdots \partial_{a_{\not D}}F_{d_{\not D}}(s) \\ &\qquad \times \frac{\big(F(\sigma)-F(s)\big)_{d_{D-1}}}{\big|F(\sigma)-F(s)\big|^D} \\ &\quad +\epsilon_{d_1\cdots d_D} \int d^{\not D} s\, \partial_{a_1}F_{d_1}(s)\cdots \partial_{a_{\not D}}F_{d_{\not D}}(s) \frac{\big(H(\sigma)-H(s)\big)_{d_{D-1}}}{\big|F(\sigma)-F(s)\big|^D} \\ &\quad -D\epsilon_{d_1\cdots d_D} \int d^{\not D} s\, \partial_{a_1}F_{d_1}(s)\cdots \partial_{a_{\not D}}F_{d_{\not D}}(s) \\ &\qquad \times \big(F(\sigma)-F(s)\big)\cdot\big(H(\sigma)-H(s)\big) \frac{\big(F(\sigma)-F(s)\big)_{d_{D-1}}}{\big|F(\sigma)-F(s)\big|^{D+2}} \\ &\quad +\mathcal{O}\left(H^2\right). \qquad \text{(F.2)} \end{aligned}$$

As $F(s)$ describes a hyper-plane, its derivatives with respect to s_a are simply Kronecker deltas: $\partial_a F_d(s) = \delta_{ad}$. Furthermore, the term of zeroth order in H on the right hand side of Eq. (F.2), viz.,

$$\begin{aligned} &\epsilon_{d_1\cdots d_D} \int d^{\not D} s\, \delta_{a_1 d_1}\cdots \delta_{a_{\not D} d_{\not D}} \frac{\big(\sigma - s\big)_{a_\nu}\delta_{a_\nu d_{D-1}}}{\big|F(\sigma)-F(s)\big|^D} \\ &\qquad\qquad = \epsilon_{a_1\cdots a_{D-2} a_\nu d_D} \int d^{\not D} s\, \frac{\big(\sigma - s\big)_{a_\nu}}{\big|F(\sigma)-F(s)\big|^D}, \end{aligned}$$

vanishes because a_ν can only take values between a_1 and $a_{\not D}$ and the term is already skew-symmetric in these indices. The last term of first order in H on the right hand side of Eq. (F.2) also vanishes, via the orthogonality of F and H.

We observe that the remaining terms for the velocity, i.e.,

$$U_{d_D}(\sigma) \approx \sum_{\nu,\nu'=1}^{\not{D}} \int d^{\not{D}}s\; \epsilon_{12\cdots d_\nu \cdots D-2\, d_{D-1} d_D}\, \partial_{a_\nu} H_{d_\nu}(s)\, \frac{\big(\sigma - s\big)_{\nu'}\, \delta_{\nu' d_{D-1}}}{\big|F(\sigma) - F(s)\big|^D}$$
$$+ \int d^{\not{D}}s\, \epsilon_{12\cdots D-2\, d_{D-1} d_D}\, \frac{\big(H(\sigma) - H(s)\big)_{d_{D-1}}}{\big|F(\sigma) - F(s)\big|^D}, \tag{F.3}$$

are nonzero only for the components U_{D-1} and U_D. In order to keep the notation compact, it is useful to introduce a new index, γ, which takes only the values $D-1$ and D. As a last step, we rearrange the indices of the Levi-Civita symbols, arriving at the $(D-1)^{\text{th}}$ and D^{th} components of the velocity, to first order in the height:

$$U_{\gamma'}(\sigma) \approx \sum_{\gamma=D-1}^{D} \epsilon_{\gamma\gamma'} \int \frac{d^{\not{D}}s}{\big|F(\sigma) - F(s)\big|^D}$$
$$\times \Big(\big(H(\sigma) - H(s)\big)_\gamma - \sum_{\nu=1}^{\not{D}} (\partial_\nu H_\gamma(s))(\sigma - s)_\nu \Big), \tag{F.4}$$

with all other components of U vanishing to this order.

Appendix G

Fourier transform of the velocity for the weak distortion of a hyper-planar vortex

The required Fourier transform is of the convolution on the right-hand side of Eq. (3.54), and is of the form

$$
\begin{aligned}
&\int d^{\not{D}}\sigma \ \mathrm{e}^{-iq\cdot\sigma} \int d^{\not{D}}s \, \frac{(\sigma - s)_\nu (\sigma - s)_{\bar{\nu}}}{|\sigma - \Sigma|^D} \, \frac{\partial^2 H_\gamma}{\partial s_\nu \partial s_{\bar{\nu}}} && \text{(G.1)} \\
= \quad & (2\pi)^{\not{D}} \int d^{\not{D}}\sigma \ \mathrm{e}^{-iq\cdot(\sigma - s)} \frac{(\sigma - s)_\nu (\sigma - s)_{\bar{\nu}}}{|\sigma - \Sigma|^D} \int d^{\not{D}}s \, \mathrm{e}^{-iq\cdot s} \frac{\partial^2 H_\gamma}{\partial s_\nu \partial s_{\bar{\nu}}} && \text{(G.2)} \\
= & -(2\pi)^{\not{D}} \, \widehat{H}_\gamma(q) \int d^{\not{D}}\sigma \, \mathrm{e}^{-iq\cdot\sigma} \frac{(q\cdot\sigma)^2}{|\sigma|^D}. && \text{(G.3)}
\end{aligned}
$$

The remaining integral is readily computed in hyper-spherical coordinates:

$$
\begin{aligned}
&\int d^{\not{D}}\sigma \, \mathrm{e}^{-iq\cdot\sigma} \frac{(q\cdot\sigma)^2}{|\sigma|^D} && \text{(G.4)} \\
&= |q|^2 \mathcal{A}_{\not{D}-1} \int s^{\not{D}-1} \, ds \frac{1}{s^D} \int_0^\pi d\theta \, \sin^{\not{D}-2}\theta \, (s\cos\theta)^2 \mathrm{e}^{-iqs\cos\theta} && \text{(G.5)} \\
&= |q|^2 \frac{\pi^{\not{D}/2}}{\Gamma(\not{D}/2)} \, \ln\left(1/|q|\xi\right), && \text{(G.6)}
\end{aligned}
$$

where we have imposed a short-distance cutoff at $s = \xi$ to render the integral convergent. The spatial Fourier transform evaluated in the present appendix, together with the temporal one, convert the differential (in time), integro-differential (in space) equation of motion (3.55) into algebraic form; see Eq. (3.59).

Appendix H

Vortex integration in the partition function

To compute the quantity in Eq. (4.24) we shall evaluate the following intermediate result, taking into account that the size of the vortex k is much smaller than the average distance between two vortices:

$$
\begin{aligned}
J^{(k)}_{d_1 d_2 \cdots d_{\not{D}}}(x_k)\, U(x_k - x) &= J^{(k)}_{d_1 d_2 \cdots d_{\not{D}}}(x_k)\, U(R^{(k)} + \rho^{(k)} - x) \\
&= \int d^{\not{D}} s\, \varepsilon_{a_1 \cdots a_{\not{D}}} (\partial_{a_1} \rho^{(k)}_{d_1}) \cdots (\partial_{a_{\not{D}}} \rho^{(k)}_{d_{\not{D}}})\, U(R^{(k)} + \rho^{(k)} - x) \\
&= \int d^{\not{D}} s\, \varepsilon_{a_1 \cdots a_{\not{D}}} (\partial_{a_1} \rho^{(k)}_{d_1}) \cdots (\partial_{a_{\not{D}}} \rho^{(k)}_{d_{\not{D}}}) \\
&\qquad \times \Big(U(R^{(k)} - x) + \rho^{(k)}_{d_{D-1}} \nabla_{d_{D-1}} U(R^{(k)} - x) + \cdots \Big)\,,
\end{aligned}
\tag{H.1}
$$

where $\nabla_{d_{D-1}}$ represents derivative with respect to the position of the center of the vortex k (i.e. $\nabla_d = \partial / \partial R^{(k)}_d$). The first term in the sum is zero because the $\rho^{(k)}$ is a periodic function of the parameters s; the second term is proportional to the volume of the volume of the $(D-1)$-dimensional hyper-sphere bounded by the vortex submanifold, denoted by $\mathcal{A}_{D-1}$, yielding the result:

$$
J^{(k)}_{d_1 d_2 \cdots d_{\not{D}}}(x_k)\, U(x_k - x) \approx \epsilon_{d_1 \cdots d_D} (\mathcal{A}_{D-1})_{d_D} \nabla_{d_{D-1}} U(R^{(k)} - x) + \cdots \tag{H.2}
$$

Note that the volume of a $(D-1)$-dimensional hyper-sphere embedded in a D-dimensional space is a one dimensional vector [1] pointing in the remaining direction.

Reverting to the computation of the quantity in Eq. (4.24), we will write:

$$
\begin{aligned}
\exp\Bigg[-\mathcal{C}_2 K_l \sum_{i\neq k} & J^{(i)*}_{d_1 d_2\cdots d_{\not D}}(x) U(x-x_k) J^{(k)}_{d_1 d_2\cdots d_{\not D}}(x_k)\Bigg] \\
\approx \prod_{i\neq k}\prod_{x} & \Big(1 - \mathcal{C}_2 K_l J^{(i)*}_{d_1 d_2\cdots d_{\not D}}(x) U(x-x_k) J^{(k)}_{d_1 d_2\cdots d_{\not D}}(x_k)\Big) \\
\approx \prod_{i,j\neq k}\prod_{x,x'} & \Big(1 - \mathcal{C}_2 K_l J^{(i)*}_{d_1 d_2\cdots d_{\not D}}(x) U(x-x_k) J^{(k)}_{d_1 d_2\cdots d_{\not D}}(x_k) \\
& -\mathcal{C}_2 K_l J^{(j)*}_{d_1 d_2\cdots d_{\not D}}(x') U(x'-x_k) J^{(k)}_{d_1 d_2\cdots d_{\not D}}(x_k) \\
& +\mathcal{C}_2^2 K_l^2 J^{(k)}_{d_1 d_2\cdots d_{\not D}}(x_k) U(x-x_k) \\
& \quad \times J^{(k)*}_{\bar d_1 \bar d_2\cdots \bar d_{\not D}}(x_k) U(x'-x_k) J^{(i)*}_{d_1 d_2\cdots d_{\not D}}(x) J^{(j)}_{\bar d_1 \bar d_2\cdots \bar d_{\not D}}(x')\Big) \\
\approx \prod_{i,j\neq k}\prod_{x,x'} & \Big(1 - \mathcal{C}_2 K_l \epsilon_{d_1\cdots d_D} (\mathcal{A}_{D-1})_{d_D} \nabla_{d_{D-1}} U(R^{(k)}-x) J^{(i)*}_{d_1 d_2\cdots d_{\not D}}(x') \\
& -\mathcal{C}_2 K_l \epsilon_{d_1\cdots d_D} (\mathcal{A}_{D-1})_{d_D} \nabla_{d_{D-1}} U(R^{(k)}-x') J^{(j)*}_{d_1 d_2\cdots d_{\not D}}(x') \\
& +\mathcal{C}_2^2 K_l^2 \epsilon_{d_1\cdots d_D} (\mathcal{A}_{D-1})_{d_D} \nabla_{d_{D-1}} U(R^{(k)}-x) \\
& \quad \times \epsilon_{\bar d_1\cdots \bar d_D} (\mathcal{A}_{D-1})_{\bar d_D} \nabla_{\bar d_{D-1}} U(R^{(k)}-x') \\
& \quad \times J^{(i)*}_{d_1 d_2\cdots d_{\not D}}(x) J^{(j)}_{\bar d_1 \bar d_2\cdots \bar d_{\not D}}(x')\Big)
\end{aligned}
\tag{H.3}
$$

In the partition function we will have to average over all orientations of the "moment"of the vortex. We will use the following relations

$$\langle V_d \rangle = 0 \tag{H.4}$$

$$\langle V_d V_{\bar d} \rangle = \frac{1}{D}\delta_{d\bar d}, \tag{H.5}$$

where V_d is an arbitrary vector and $\langle\rangle$ denotes angular average. The terms linear in $(\mathcal{A}_{D-1})_{d_D}$,

[1] In an analogy with electrodynamics we say that the magnetic moment of a current loop (i.e. 1-dimensional submanifold), is proportional to the area (i.e. volume of the 2-dimensional surface) of the loop and points in a direction perpendicular to the area

will hence vanish when taking the average, and we obtain:

$$\prod_{i,j\neq k}\prod_{x,x'}\Bigg(1+\frac{1}{D}\mathcal{C}_2^2K_l^2(\mathcal{A}_{D-1})^2\epsilon_{d_1\cdots d_{D-1}d_D}\epsilon_{\bar{d}_1\cdots\bar{d}_{D-1}d_D}$$
$$\times J^{(i)*}_{d_1d_2\cdots d_{\not{D}}}(x)\nabla_{d_{D-1}}U(R^{(k)}-x)\nabla_{\bar{d}_{D-1}}U(R^{(k)}-x')J^{(j)}_{\bar{d}_1\bar{d}_2\cdots\bar{d}_{\not{D}}}(x')\Bigg). \tag{H.6}$$

The contraction of the two skew-symmetric symbols $\epsilon_{d_1\cdots d_{D-1}d_D}\epsilon_{\bar{d}_1\cdots\bar{d}_{D-1}d_D}$ contains two types of terms:

(a) $(D-2)!$ Kronecker delta terms that pair up the indices d_{D-1} and $\bar{d}_{D-1}$, (i.e. $\cdots\delta_{d_{D-1}\bar{d}_{D-1}}$). Note that each product of δ terms is multiplied by the signature of the permutation of the indices $d_1\cdots d_{\not{D}}$. Using the fact that the source currents are antisymmetric tensors in their indices, we conclude that all these $\not{D}$ terms are positive.

(b) $(D-2)(D-2)!$ terms that mix up the indices d_{D-1} and $\bar{d}_{D-1}$, (i.e. $\cdots\delta_{i\bar{d}_{D-1}}\delta_{\bar{i}d_{D-1}}$, where i and $\bar{i}$ are indices belonging to the sets $\{d_1,\cdots d_{\not{D}}\}$, respectively $\{\bar{d}_1,\cdots\bar{d}_{\not{D}}\}$)

We shall argue that that the angular average of the (b) terms is zero, by arbitrarily choosing to compute one of them:

$$\begin{aligned}
&\int\frac{d^DR_k}{a^D}J^{(i)*}_{d_1d_2\cdots d_{\not{D}}}(x)\nabla_{d_{D-1}}U(R^{(k)}-x)\nabla_{d_{D-2}}U(R^{(k)}-x')J^{(j)}_{d_1\cdots d_{D-3}d_{D-1}}(x')\\
&=\int\frac{d^DR_k}{a^D}J^{(i)*}_{d_1d_2\cdots d_{\not{D}}}(x)J^{(j)}_{d_1\cdots d_{D-3}d_{D-1}}(x')\\
&\quad\times\Bigg[\nabla_{d_{D-2}}\left(\nabla_{d_{D-1}}U(R^{(k)}-x)U(R^{(k)}-x')\right)\\
&\qquad+\nabla_{d_{D-2}}\left(\nabla_{d_{D-1}}U(R^{(k)}-x)\right)U(R^{(k)}-x')\Bigg]\\
&=\int\frac{d^DR_k}{a^D}J^{(i)*}_{d_1d_2\cdots d_{\not{D}}}(x)J^{(j)}_{d_1\cdots d_{D-3}d_{D-1}}(x')\frac{(D-1)(D-2)}{|R^{(k)}-x'|^{\not{D}}}\\
&\quad\times\frac{R^{(k)}_{d_{D-1}}R^{(k)}_{d_{D-2}}-R^{(k)}_{d_{D-1}}x_{d_{D-2}}-x_{d_{D-1}}R^{(k)}_{d_{D-2}}+x_{d_{D-1}}x_{d_{D-2}}}{|R^{(k)}-x|^D},
\end{aligned} \tag{H.7}$$

where we used the fact that the total derivative term vanishes, due to periodic boundary conditions on the volume $\Omega=\int d^DR_k$. By virtue of Eq. (H.5) the angular averages $\langle R^{(k)}_{d_{D-1}}R^{(k)}_{d_{D-2}}\rangle$,

$\langle R^{(k)}_{d_{D-1}} x_{d_{D-2}} \rangle$, $\langle x_{d_{D-1}} R^{(k)}_{d_{D-2}} \rangle$, and $\langle x_{d_{D-1}} x_{d_{D-2}} \rangle$ are zero, therefore all the terms of (b) type vanish.

Taking into account the integral over the size of the vortex

$$\mathcal{A}_{\not{D}-1} \int_a^{a+da} \frac{d\rho}{a} = \mathcal{A}_{\not{D}-1} dl \tag{H.8}$$

the final result for quantity $\mathcal{J}$ in Eq. (4.24) is:

$$\begin{aligned} \mathcal{J} \approx \int \frac{d^D R_k}{a^D} \mathcal{A}_{\not{D}-1} y_l dl \prod_{i,j\neq k} \prod_{x,x'} \Bigg(1 + \frac{\not{D}!}{D} \mathcal{C}_2^2 K_l^2 (\mathcal{A}_{D-1})^2 \\ \times J^{(i)*}_{d_1 d_2 \cdots d_{\not{D}}}(x) \nabla_{d_{D-1}} U(R^{(k)} - x) \nabla_{d_{D-1}} U(R^{(k)} - x') J^{(j)}_{d_1 d_2 \cdots d_{\not{D}}}(x') \Bigg). \end{aligned} \tag{H.9}$$

References

[1] The term $\mathcal{A}_\Delta$ is defined to be the area of the $(\Delta - 1)$-dimensional surface of a Δ-dimensional hyper-sphere of unit radius, viz., $\mathcal{A}_\Delta = 2\Gamma(1/2)^\Delta / \Gamma(\Delta/2)$, where $\Gamma(z)$ is the standard Gamma function [25].

[2] D. J. Bishop and J. D. Reppy. Study of the superfluid transition in two-dimensional he^4 films. *Phys. Rev. B*, 22(11):5171–5185, Dec 1980.

[3] J. Cardy. *Scaling and renormalization in statistical Physics.* Cambridge Lecture Notes in Physics, 2000.

[4] Biplab Chattopadhyay, Mangal C. Mahato, and Subodh R. Shenoy. Vortex-loop crinkling in the three-dimensional xy model: Numerical evidence in support of an ansatz. *Phys. Rev. B*, 47(22):15159–15169, Jun 1993.

[5] Biplab Chattopadhyay and Subodh R. Shenoy. Kosterlitz-thouless signatures from 3d vortex loops in layered superconductors. *Phys. Rev. Lett.*, 72(3):400–403, Jan 1994.

[6] R. J. Donnelly. *Quantized Vortices in Helium II.* Cambridge University Press, 1991.

[7] L. M. Brown (editor). *Selected Papers of Richard Feynman.* World Scientific, 2000.

[8] P. M. Goldbart F. Bora. Quantized vortices and superflow in arbitrary dimensions: structure, energetics and dynamics. *Journal of Physics A: Mathematical and Theoretical*, 42(18):185001 (30pp), 2009.

[9] R. P. Feynman. Application of quantum mechanics to liquid helium. *Progress in Low Temperature Physics*, 1:17–53, 1954.

[10] E. Fradkin. *Field Theories of Condensed Matter Systems.* Westview, 1991.

[11] T. Frankel. *The Geometry of Physics: An Introduction.* Cambridge University Press, 2003.

[12] F. Bora G. Williams, P. M. Goldbart. Generalize vortex-loop approach to the superfluid transition in arbitrary dimensions. *In preparation*, 2010.

[13] N. Goldenfeld. *Lectures on phase transitions and the renormalization group.* Westview, 1992.

[14] M. Henneaux and C. Teitelboim. p-form electrodynamics. *Foundations of Physics*, 16(7):593 – 617, July 1986.

[15] Jorge V. José, Leo P. Kadanoff, Scott Kirkpatrick, and David R. Nelson. Renormalization, vortices, and symmetry-breaking perturbations in the two-dimensional planar model. *Phys. Rev. B*, 16(3):1217–1241, Aug 1977.

[16] M. Kalb and P. Ramond. Classical direct interstring action. *Phys. Rev. D*, 9(8):2273–2284, April 1974.

[17] H. Kleinert. *Gauge Fields in Condensed Matter*. World Scientific, 1989.

[18] J M Kosterlitz and D J Thouless. Ordering, metastability and phase transitions in two-dimensional systems. *Journal of Physics C: Solid State Physics*, 6(7):1181–1203, 1973.

[19] J M Kosterlitz and D J Thouless. Ordering, metastability and phase transitions in two-dimensional systems. *Journal of Physics C: Solid State Physics*, 6(7):1181–1203, 1973.

[20] L. Landau. Theory of the superfluidity of helium ii. *Phys. Rev.*, 60(4):356–358, Aug 1941.

[21] L. D. Landau and E. M. Lifshitz. *Statistical Physics II*. Pergamon Press, 1980.

[22] A. J. Leggett. Lecture notes: Physics in two dimensions.

[23] J. A. Lipa and T. C. P. Chui. Very high-resolution heat-capacity measurements near the lambda point of helium. *Phys. Rev. Lett.*, 51(25):2291–2294, Dec 1983.

[24] F. London. On the bose-einstein condensation. *Phys. Rev.*, 54(11):947–954, Dec 1938.

[25] I. Stegun M. Abramowitz. *Handbook of Mathematical Functions*. Dover, 1964. Gamma function and related functions.

[26] S-K Ma. *Modern theory of critical phenomena*. Westview, 2000.

[27] N. D. Mermin. The topological theory of defects in ordered media. *Rev. Mod. Phys.*

[28] David R. Nelson and Daniel S. Fisher. Dynamics of classical xy spins in one and two dimensions. *Phys. Rev. B*, 16(11):4945–4955, Dec 1977.

[29] David R. Nelson and J. M. Kosterlitz. Universal jump in the superfluid density of two-dimensional superfluids. *Phys. Rev. Lett.*, 39(19):1201–1205, Nov 1977.

[30] L. Onsager. Discussione e osservazioni. *Nuovo Cimento*, 6:249–250, 1949.

[31] T. C. Lubensky P. M. Chaikin. *Principles of Condensed Matter Physics*. Cambridge University Press, 2000.

[32] A. Pressley. *Elementary Differential Geometry.* Springer, 2001.

[33] M. Rasetti and T. Regge. Vortices in he ii, current algebras and quantum knots. *Physica A: Statistical and Theoretical Physics*, 80(3):217 – 233, 1975.

[34] Joseph Rudnick and David R. Nelson. Equations of state and renormalization-group recursion relations. *Phys. Rev. B*, 13(5):2208–2221, Mar 1976.

[35] B. F. Schutz. *Geometrical Methods of Mathematical Physics.* Cambridge University Press, 1980.

[36] Subodh R. Shenoy. Vortex-loop scaling in the three-dimensional xy ferromagnet. *Phys. Rev. B*, 40(7):5056–5068, Sep 1989.

[37] C. Teitelboim. Gauge invariance for extended objects. *Physics Letters B*, 167(1):63 – 68, 1986.

[38] W. Thomson. On vortex atoms. *Proc. Royal Soc. Edinburgh*, VI:94–105, 1867.

[39] W. Thomson. Vibrations of a columnar vortex. *Philosophical Magazine Series 5*, 10:155–168, 1880.

[40] D. J. Thouless. *Topological Quantum Numbers in Nonrelativistic Physics.* World Scientific, 1998.

[41] Gary A. Williams. Vortex-ring model of the superfluid λ transition. *Phys. Rev. Lett.*, 59(17):1926–1929, Oct 1987.

[42] Gary A. Williams. Vortex dynamics at the superfluid λ-transition. *Journal of Low Temperature Physics*, 93(5):1079–1095, 1993.

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